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Π^1_2 SETS AND Π^1_2 SINGLETONS

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ABSTRACT. The following are equivalent:

(a) every real is constructible;

(b) every nonempty Π^1_2 set of reals contains a Π^1_2 singleton.

(Implication (a) $\Rightarrow$ (b) is due solely to H. Friedman.)

We refer the reader to [6] and [9] for the pertinent notation and definitions.

One of the most fruitful activities in effective descriptive set theory has been the computing of bases for various levels of the projective hierarchy. (See [4] for a summary and for the most recent developments.) The central result is undoubtably the Kondo-Addison theorem: *A nonempty* Π^1_1 (a) *set of reals contains a* Π^1_1 (a) *singleton, uniformly in* a (*for reals* a). This establishes *uniformization* for Π^1_1. Uniformization for Σ^1_2 is an immediate consequence of this. Since uniformization implies reduction, uniformization cannot hold on both sides of a level of the projective hierarchy (see [2] for details). Hence uniformization for Π^1_2 fails.

But what if we drop the uniformity from uniformization? As H. Friedman has shown, if every real is constructible, then every nonempty Π^1_2 set of reals contains a Π^1_2 singleton. (Friedman has actually established something more general—see Theorem 1 below.)[1] Our main result[2] is the converse of this, namely: If there is a nonconstructible real then there is a nonempty Π^1_2 set of reals which contains no Π^1_2 singleton.

Let R = the set of reals, where a real is a subset of ω = the integers. Let $a, b, c, \cdots$ denote reals. By the usual standard devices, code a pair of reals or a countable set of reals by one single real.

Definition [10]. A well ordering $<$ of R is *strongly* Δ^1_n if: $<$ is Δ^1_n, $<$ has order type $\aleph_1$, and $\{a \in R: a$ codes an initial segment of $<\}$ is Σ^1_n.

The main feature of a strongly Δ^1_n well ordering, $<$, is that quantification over $\{b: b < c\}$ is transformed (in a Σ^1_n (c) way uniformly in c) into number quantification (by using a real which codes $\{b: b < c\}$). Thus such quantification will not increase a hierarchy calculation. Notice in addition that each

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[1] We wish to thank H. Friedman for consenting to the inclusion of the proof of his result in this paper.

[2] We would also like to commend Harvey Friedman for persistently affirming his belief in this result—despite our equally persistent assurance otherwise.

real $< c$ is $\Delta_n^1(c)$ (since the first (wrt $<$) real which codes $\{b: b < c\}$ is $\Delta_n^1(c)$).

Theorem 1 (H. Friedman). *If there is a strongly Δ_n^1 well ordering of R, then every nonempty Π_n^1 set of reals contains a Π_n^1 singleton.*

Proof. Let $<$ be a strongly Δ_n^1 well ordering of R. We recall a convenient definition in this context. (The following definition is slightly nonstandard. It can be made standard by replacing the real b with the ordinal: the order type of $\{c: c < b\}$.)

Definition. $b \in R$ is called *stable* if: for all $d < b$ and all Π_{n-1}^1 formulas $\theta(x, y)$, $(\exists x)\theta(x, d) \Rightarrow (\exists x < b)\theta(x, d)$.

For a real c, $\delta_n^1(c)$ = the first (wrt $<$) stable real $> c$.

Notice that for reals c, a, a is $\Delta_n^1(c)$ iff $a < \delta_n^1(c)$. [Proof. (This is folklore.) If a is $\Delta_n^1(c)$ then there is a Π_{n-1}^1 formula $\theta(w, x, y)$ such that $(\exists w)\theta(w, x, c) \Leftrightarrow x = a$. Thus for some w, $\langle w, a\rangle < \delta_n^1(c)$ and hence $a < \delta_n^1(c)$. For the other direction, let b = the first (wrt $<$) real not $\Delta_n^1(c)$. Given $d < b$ and given a Π_{n-1}^1 formula $\theta(x, y)$, if $(\exists x)\theta(x, d)$ then the first (wrt $<$) such x is $\Delta_n^1(d)$ and hence $x < b$. Thus b is stable, so $b = \delta_n^1(c)$. Hence $a < \delta_n^1(c) = b \Rightarrow a$ is $\Delta_n^1(c)$.]

Now let Y be a nonempty Π_n^1 set of reals. Let a = the first (wrt $<$) real in Y. We claim that $\{a\}$ is Π_n^1.

For some Σ_{n-1}^1 formula $\psi(x, y)$ we have: $y \in Y$ iff $\forall x\psi(x, y)$. Let $W = \{b: (\forall c < b)(\exists d < b)\neg\psi(d, c)\}$. W is Δ_n^1. Notice that if b is stable and if $b \leq a$, then $b \in W$. Also notice that $b \in W \Rightarrow b \leq a$.

Let c = the first (wrt $<$) real which is $\geq b$ for all $b \in W$. An examination of the definition of W reveals that $c \in W$. Clearly $c \leq a$. c is uniformly $\Delta_n^1(d)$ for all $d \geq a$; that is, there is a Σ_n^1 formula $\phi(z, x)$ such that: for all $d \geq a$, $\phi(z, d) \Leftrightarrow z = c$. $\phi(z, x)$ simply says: $z \in W$ and $(\forall w \leq x)(w \in W \Rightarrow w \leq z)$.

Now, $\delta_n^1(c) > c$, and thus $\delta_n^1(c) \notin W$. But $\delta_n^1(c)$ is stable. Hence $a < \delta_n^1(c)$, and so a is $\Delta_n^1(c)$. Thus there is a Σ_n^1 formula $\theta(y, z)$ such that $\theta(y, c) \Leftrightarrow y = a$.

But now we have: $x = a$ iff $x \in Y$ and $\forall z\forall y\,[(\phi(z, x)$ and $\theta(y, z)) \Rightarrow y = x]$, which is Π_n^1. □

We should point out that the Π_n^1 singletons produced by the above proof are of a rather different species than Π_1^1 singletons. It is well known that a real of the same Δ_1^1 degree as a Π_1^1 singleton is also a Π_1^1 singleton. The corresponding fact is not true for Π_n^1 singletons under the hypothesis of the above theorem. For example, let a be the first (wrt $<$) real such that: for each Π_n^1 formula $\phi(x)$, if $\phi(a)$ then $\exists x(\phi(x)$ and x is *strictly* Δ_n^1 in a). a is not a Π_n^1 singleton, but it can be verified that a has the same Δ_n^1 degree as a Π_n^1 singleton.

The constructible reals are known (see [1]) to have a strongly Δ^1_2 well ordering. Thus

Corollary. *If every real is constructible, then every nonempty* Π^1_n, $n \geq 2$, *set of reals contains a* Π^1_n *singleton.* □

Theorem 2. *There is a* Π^1_2 *predicate* $\phi(x)$ *such that* ZF $\vdash \exists x \phi(x)$, *and* ZF + *"there is a nonconstructible real"* $\vdash \forall x(\phi(x) \Rightarrow x$ *is not a* Π^1_2 *singleton*).

Proof. Intuitively, $\phi(x)$ will assert (among other things) that x is a member of every reasonably large Σ^1_2 set of reals, thus making it impossible for $R \sim \{x\}$ to be Σ^1_2 (unless of course every real is constructible, in which case $R \sim \{x\}$ just is not large enough). We now express this assertion in a Π^1_2 way.

A real is a subset of ω, and hence may be viewed as an infinite sequence of 0's and 1's. Let $B = 2^{<\omega}$ = the full binary tree = the set of all finite sequences of 0's and 1's. For $s, t \in B$, $s \subseteq t$ if t extends s; $s|t$ if s and t have no common extension; $l(s)$ = the length of s. A *tree* T is a subset of B such that: $t \in T$, $s \subseteq t \Rightarrow s \in T$; $s \in T \Rightarrow (\exists t, u \in T)(s \subseteq t, u$ and $t|u)$. A branch through T is a real b such that every initial segment of b is in T; $[T]$ is the set of all branches through T. For $s \in T$, let $T_s = \{t \in T: s \subseteq t$ or $t \subseteq s\}$.

We will use the following enumeration of Σ^1_2 sets: let $\phi(e, x)$ be a universal Σ_1 formula of the language of set theory; the eth Σ^1_2 set of reals, $A(e)$, is defined by, $x \in A(e)$ iff $L[x] \models \phi(e, x)$. (This does enumerate all Σ^1_2 sets of reals. See [8].) For an ordinal α, let $A(e, \alpha) = \{x \in R: L_\alpha[x] \models \phi(e, x)\}$. (If α is countable then $A(e, \alpha)$ is Borel. This is nothing more than the usual decomposition of a Σ^1_2 set into a union of $\aleph_1$ Borel sets.) If T is a tree, then the assertion: "$[T] \subseteq A(e, \alpha)$" is Π^1_1 over the structure $L_\alpha[T]$, and hence is absolute for countable α's. We will use without explicit mention the fact that similar statements are also absolute.

We will need the following result of Solovay [11] and Mansfield [5]:

The perfect set theorem. *If* T *is a tree in* L, *and if* $[T] \cap A(e)$ *contains a nonconstructable real, then for some tree* $T' \subseteq T$ *and some* $\alpha < \aleph^L_1$, $[T'] \subseteq A(e, \alpha)$.

We now define, for all ordinals $j \leq \lambda$ (where λ will eventually turn out to be the ordinal δ^1_2), an ordinal $\alpha(j)$, a set of integers $Z(j)$, and a set of trees $X(j)$. The map $j \mapsto \alpha(j)$, $Z(j)$, $X(j)$ will be Σ_1 over $L_{\aleph^L_1}$. (So in particular each tree in $X(j)$ is constructible.) In addition, the following properties will hold.

For $j \leq \lambda$ and for $T \in X(j)$:

(i) For all $i < j$, $\alpha(i) < \alpha(j)$ and $Z(i) \subset Z(j)$.

(ii) $s \in T \Rightarrow T_s \in X(j)$; for all $i < j$ and all $S \in X(i)$, $(\exists S' \in X(j))$ $(S' \subseteq S)$.

(iii) For all $i < j$, $[T] \subseteq \bigcup\{[S]: S \in X(i)\}$.

(iv) $e \in Z(j) \Rightarrow [T] \subseteq A(e, \alpha(j))$.

The following should be reminiscent of Jensen's construction in [3].

Case 1. $j = 0$. Let $\alpha(0) = 0$, $Z(0) = \varnothing$, $X(0) = \{B_s: s \in B\}$.

Case 2. j a limit ordinal. We are given $\alpha(i)$, $Z(i)$, $X(i)$ for $i < j$. We claim that for each $k < j$ and each $U \in X(k)$, there is a tree $T \subseteq U$ such that T satisfies (iii) above. [T is found by fusion, as in [7] or [3, p. 124]. Let $k = i_0 < i_1 < \cdots$ be such that $U_n i_n = j$. Using (ii) above, find for each $t \in B$ a tree T^t such that:

$T^{\varnothing} = U$, $l(t) = n \Rightarrow T^t \in X(i_n)$,

$u \subseteq t \Rightarrow T^t \subseteq T^u$,

$u|t \Rightarrow [T^u] \cap [T^t] = \varnothing$, and

$l(t) = n \Rightarrow T^t = (T^t)_s$ where $l(s) = n$.

Let $T = \bigcap_n \bigcup\{T^t: t \in B$ and $l(t) = n\}$. It can be checked that T is a tree. Clearly $T \subseteq T^{\varnothing} = U$. For each n, $[T] \subseteq \bigcup\{[T^t]: l(t) = n\}$. Thus $[T] \subseteq \bigcup\{[S]: S \in X(i_n)\}$. So T satisfies (iii).]

For each $k < j$ and each $U \in X(k)$, let U' = the first (first in the sense of L) tree $T \subseteq U$ such that T satisfies (iii).

Let $X(j) = \{U'_s: U \in X(k)$ for some $k < j$, and $s \in U'\}$. So (ii) and (iii) are satisfied. Let $\alpha(j) = U_{i<j}\alpha(i)$, and let $Z(j) = U_{i<j}Z(i)$. So (i) holds. Since (iii) holds, so does (iv).

Case 3. $j = i + 1$. (This is the only interesting case.) We are given $\alpha(i)$, $Z(i)$, $X(i)$. Let $\alpha(j)$ = the first $\alpha > \alpha(i)$, $\alpha < \aleph_1^L$, for which there is $e \notin Z(i)$ such that $(\forall T \in X(i))(\exists$ a tree $T' \subseteq T$, $T' \in L_\alpha)([T'] \subseteq A(e, \alpha))$. [If there is no such α, then we let $\lambda = i$, and we leave $\alpha(j)$, $Z(j)$, $X(j)$ undefined.] Let e_0 be the first such e. Let $Z(j) = Z(i) \cup \{e_0\}$. For each $T \in X(i)$, let T' = the first (first in the sense of L) tree such that $T' \subseteq T$ and $[T'] \subseteq A(e_0, \alpha(j))$. Let $X(j) = \{T'_s: T \in X(i)$ and $s \in T'\}$. Clearly (i) through (iv) are satisfied.

This completes the definition of $\alpha(j)$, $Z(j)$, $X(j)$, $j \leq \lambda$.

Remarks. (1). It is easy to see that the above construction continues for at least δ_2^1 steps (i.e., if $i < \delta_2^1$ then $\alpha(i + 1)$ is defined). But $Z(\lambda)$ is clearly a Σ_2^1 set of integers, and hence $\lambda = \delta_2^1$.

(2). If $e \notin Z(\lambda)$, then (since $\alpha(\lambda + 1)$ is undefined) there is $T \in X(\lambda)$ such that for all trees $T' \subseteq T$ and all $\alpha < \aleph_1^L$, $[T'] \nsubseteq A(e, \alpha)$.

We can now produce the promised Π_2^1 predicate ϕ. For $x \in R$, let

$$\phi(x) \equiv (\forall j \leq \lambda)(\forall e \in Z(j))(x \in A(e, \alpha(j))).$$

Since the map $j \mapsto \alpha(j)$, $Z(j)$ is Σ_1 over $L_{\aleph_1}$, ϕ is Π_2^1.

By (iii) and (iv), every branch through a member of $X(\lambda)$ is a solution of ϕ. Thus ZF $\vdash \exists x \phi(x)$. (Notice that ϕ is actually Δ_1^1 in any real which enumerates λ.)

Now suppose we are in a model of ZF, and we have a real b such that $\phi(b)$ and such that $\{b\}$ is Π^1_2. For some e, $\{b\} = R \sim A(e)$. Since $b \notin A(e)$, $e \notin Z(\lambda)$. Thus there is $T \in Z(\lambda)$ as described in Remark (2). But then by the perfect set theorem mentioned earlier, every nonconstructible branch through T is in $R \sim A(e)$. Thus $[T]$ has at most one nonconstructible member. Since $T \in L$, this means that there are no nonconstructible reals. □

We should mention in closing that Theorem 2 seems optimal. By [3] there are models of ZFC with nonconstuctible reals in which every nonempty Π^1_2 set contains a Δ^1_3 real.

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