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FIXED POINTS OF ANTITONE MAPPINGS

ZAHAVA SHMUELY

ABSTRACT. We present a family of antitone mappings defined on complete atomic lattices which have the fixed point property. Two commuting mappings of the family have a common fixed point. An example is given of three commuting mappings of the family which do not have a common fixed point.

1. Tarski [4], Abian and Brown [1], and others have studied fixed points of isotone mappings on partially ordered sets. In [2] fixed points of certain antitone mappings are studied. Here we are dealing with a family of symmetric polarized mappings (see definitions below) which have the fixed point property. Basically, these mappings imitate the behaviour of the centralizer operator acting on the subsets of a given groupoid.

In this note A denotes a complete lattice unless otherwise stated. An antitone mapping $T: A \to A$ is *polarized* if there exists an (necessarily unique) antitone mapping denoted by $T^*: A \to A$ such that $\langle T, T^*\rangle$ is a Galois connection on A, i.e., $TT^*(p) \geq p$ and $T^*T(p) \geq p$ for each $p \in A$. $L(A)$ is the set of polarized mappings $T: A \to A$. As is well known [3], $T \in L(A)$ iff

(1) $T(0) = 1$ where 0, 1 are the universal bounds of A, and

(2) $T(\bigwedge_{p \in A'} p) = \bigwedge_{p \in A'} T(p)$ for each $A' \subseteq A$.

$T \in L(A)$ is *symmetric* if $T = T^*$. For $T \in L(A)$, let $P_T(A)$ denote the family of subsets $A_i \subseteq A$ satisfying $T(\bigvee_{p \in A_i} p) \geq \bigvee_{p \in A_i} p$. Notice that $\{0\} \in P_T(A)$. $P_T(A)$ is ordered by set inclusion. We now prove

Lemma 1. *Let A be a complete lattice and let $T \in L(A)$. Then $P_T(A)$ contains a maximal element.*

Proof. Let $\{A_j\}_{j \in J}$ be a chain in $P_T(A)$, and put $\bigvee_{p \in A_j} p = p_j$. We now show that $A_0 = \bigcup_{j \in J} A_j$ belongs to $P_T(A)$. Given any $i, j \in J$ we may assume $A_i \subseteq A_j$, and so $p_i \leq p_j \leq T(p_j) \leq T(p_i)$. It follows that $\bigvee_{i \in J} p_i \leq T(p_j)$ for each $j \in J$ and so $\bigvee_{i \in J} p_i \leq \bigwedge_{j \in J} T(p_j)$. By (2),

$$\bigvee_{p \in A_0} p = \bigvee_{i \in J} p_i \leq T\left(\bigvee_{j \in J} p_j\right) = T\left(\bigvee_{p \in A_0} p\right)$$

follows implying that $A_0 \in P_T(A)$. By Zorn's lemma $P_T(A)$ contains a maximal element.

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Theorem 1. *Let* A *be a complete atomic lattice and let* $T: A \to A$ *be a symmetric polarized mapping satisfying* $T(q) \geq q$ *for each atom* $q \in A$. *Then* T *has a fixed point.*

Proof. Let $A_0 \subseteq A$ be a maximal element of $P_T(A)$ (see Lemma 1), and put $p_0 = \bigvee_{p \in A_0} p$. Obviously, $T(p_0) \geq p_0$. Assuming $T(p_0) > p_0$ we can find an atom $q \in A$, $q \notin A_0$ such that $T(p_0) \geq q$. Since $T = T^*$, $T(q) \geq p_0$ follows. This together with $T(q) \geq q$ yields $T(p_0 \vee q) = T(p_0) \wedge T(q) \geq p_0 \vee q$, contradicting the maximality of A_0. Thus $T(p_0) = p_0$ follows.

2. One is naturally led to consider commuting mappings having the properties specified in Theorem 1. Firstly, we have

Lemma 2. *Let* A *be a partially ordered set and let* $T, S: A \to A$ *be symmetric polarized mappings. If* $TS = ST$ *then* $T^2 = S^2$.

Proof. Since $S = S^*$, $T = T^*$ it follows that $T^2(p) \geq p$, $S^2(p) \geq p$ for each $p \in A$. Hence $TST(p) = T^2S(p) \geq S(p)$. Since S is antitone $TST(p) = ST^2(p) \leq S(p)$. Thus $TST = S$ implying that $T^2S^2 = S^2$. Similarly, we have $T^2S^2 = T^2$ and $T^2 = S^2$ follows.

We can now prove

Theorem 2. *Let* A *be a complete atomic lattice and let* $T, S: A \to A$ *be symmetric polarized mappings satisfying* $T(q) \geq q$, $S(q) \geq q$ *for each atom* $q \in A$. *If* $TS = ST$ *then* T *and* S *possess a common fixed point.*

Proof. The mapping $T \wedge S: A \to A$ defined by $(T \wedge S)(p) = T(p) \wedge S(p)$ is easily shown to satisfy $T \wedge S \in L(A)$ with $(T \wedge S)^* = T \wedge S$. Since $(T \wedge S)(q) \geq q$ for each atom $q \in A$, we may apply Theorem 1 to find $p_0 \in A$ such that $(T \wedge S)(p_0) = p_0$. Since $T(p_0) \geq p_0$, $S(p_0) \geq p_0$ we get $ST(p_0) \leq S(p_0)$, $TS(p_0) \leq T(p_0)$ implying $ST(p_0) \leq S(p_0) \wedge T(p_0) = p_0$. Since $S = S^*$, $S = S^3$ follows. By Lemma 2, $T^2 = S^2$ holds. Thus, applying T on the last inclusion we get

$$S(p_0) = S^3(p_0) = T^2S(p_0) = TST(p_0) \geq T(p_0).$$

$T(p_0) \geq S(p_0)$ follows similarly. Hence $T(p_0) = S(p_0) = (T \wedge S)(p_0) = p_0$, completing the proof.

The following example is brought to show that three commuting mappings, each satisfying the conditions of Theorem 1, need not have a common fixed point.

Example. $A = 2^M$ is the complete atomic Boolean lattice of subsets of $M = \{1, 2, 3, 4, 5, 6\}$. For ease of notation let m stand for the one-element set $\{m\}$ and let p' denote the complement of $p \in A$. The polarized mappings $R, S, T: A \to A$ are given by specifying their values on the atoms of A (see

(1), (2)) as follows:

$$R: 1 \to 2', \quad 2 \to 1', \quad 3 \to 4', \quad 4 \to 3', \quad 5 \to 6', \quad 6 \to 5'.$$
$$S: 1 \to 2', \quad 2 \to 1', \quad 3 \to 5', \quad 4 \to 6', \quad 5 \to 3', \quad 6 \to 4'.$$
$$T: 1 \to 2', \quad 2 \to 1', \quad 3 \to 6', \quad 4 \to 5', \quad 5 \to 4', \quad 6 \to 3'.$$

Obviously, R, S, T are symmetric and satisfy $R(q) \geq q$, $S(q) \geq q$, $T(q) \geq q$ for each atom $q \in A$. By direct computation one can show that R, S, T form a commutative family. The fixed points of R are exactly those three-element subsets of M with each element belonging to a different cycle of the permutation which naturally corresponds to R (namely (12) (34) (56)). A similar result holds for S and T. An inspection of the corresponding cycles shows that R, S, T do not possess a common fixed point.

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DEPARTMENT OF MATHEMATICS, TECHNION, ISRAEL INSTITUTE OF TECHNOLOGY, HAIFA, ISRAEL