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ON k-FREE INTEGERS WITH SMALL PRIME FACTORS

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ABSTRACT. The object of this note is to give a nontrivial lower estimate for the function $\psi_k(x, y; h)$, defined to be the number of k-free integers m such that $1 \leq m < x$, $(m, h) = 1$, and m has no prime factor greater than or equal to y.

Let $k \geq 2$ be a fixed natural number and let $\mu_k(n)$ denote the multiplicative function given for powers of an arbitrary prime p by

$$\mu_k(p^a) = \begin{cases} 1, & a = 0, \\ -1, & a = k, \\ 0, & \text{otherwise}. \end{cases} \tag{1}$$

Let h be a natural number, $x > 0$, $y \geq 2$ be real numbers and let $f_k(n) = \sum_{d|n} \mu_k(d)$. Let $p(n)$ denote the largest prime factor of n with $p(1) = 1$. Then the sum

$$\psi_k(x, y; h) = \sum_{n<x;\, p(n)<y;\, (n,h)=1} f_k(n) \tag{2}$$

denotes the number of k-free (having no kth power divisors) natural numbers less than x, relatively prime to h, and free of prime factors greater than or equal to y.

Now we let t be a real number such that $x = y^t$ and $t = \log x/\log y$. The purpose of this note is to present a lower bound for $\psi_k(y^t, y; h)$, using only elementary methods, which has very few restrictions on y, t, and h.

Theorem. *Let $t \geq 3$. Given t, there exists a real number $y_1 = y_1(t)$ such that*

$$\psi_k(y^t, y; h) \geq 2e^{-10} \prod_{p|h} \left(\frac{1 - p^{-1}}{1 - p^{-k}}\right) \frac{y^t}{\zeta(k)} \exp\{-t(\log t + \log\log t + \eta(t))\} \tag{3}$$

for each $y \geq y_1$ and $h \leq (\log y)^C$ for some absolute constant C, where

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(4) $$\eta(t) = \frac{1}{\log t}\left(\log\log t + 1 - \frac{\log\log t}{\log t} + \frac{4}{\log t} + \frac{4\log t}{t}\right)$$

and $\zeta(k)$ *is Riemann's zeta function.*

For comparison, we note that as a consequence of the remark on p. 199 and Theorem 3.2.4 of Levin and Fainleib [5]

(5) $$\lim_{y\to\infty} \frac{\psi_k(y^t, y; h)}{\prod_{p|h}\left(\frac{1-p^{-1}}{1-p^{-k}}\right)\frac{y^t}{\zeta(k)} Z(t)} = 1$$

for $t \leq (\log y)^{3/5-\delta}$, $\delta > 0$, $h \leq (\log y)^C$, C an absolute constant, where $Z(t)$ satisfies de Bruijn's differential-difference equation $tZ'(t) = -Z(t-1)$ with initial condition $Z(t) = 1$ for $0 \leq t \leq 1$. In particular,

$$Z(t) = \exp\{-t(\log t + \log\log t + o(1))\}$$

as t approaches ∞.

We can also state a comparable upper bound with the restricted range, $e < t < y/e \log y$, $h \leq y$, using the proof of Theorem 3 of [4] with $g(n) = f_k(n)/n$:

(6) $$\psi_k(y^t, y; h) \leq \prod_{p|h}\left(\frac{1-p^{-1}}{1-p^{-k}}\right)\prod_{p<y}(1-p^{-k})y^t \exp\{-t\log t - t\log\log t + \eta(t, y)\}$$

where

(7) $$\eta(t, y) = t\left\{1 - \frac{\log\log t}{\log t} + O\left(\frac{1}{\log t}\right)\right\} + O(\log\log y) + O\left(\frac{(t\log t)^2}{y\log y}\right).$$

Throughout the discussion, the constants implied by the use of the O-notation are absolute.

In order to prove the Theorem, we use essentially the method of Halberstam [1] and [3] together with an estimate for $Q_k(x; h)$, the number of k-free natural numbers less than x that are relatively prime to h, and a generalization of the Buchstab identity.

We note that

(8) $$Q_k(x; h) = \frac{x}{\zeta(k)}\prod_{p|h}\left(\frac{1-p^{-1}}{1-p^{-k}}\right) + O\left(x^{1/k}\frac{\phi(h)}{h}\; 2^{\nu(h)}\right) + O(4^{\nu(h)})$$

where $\phi(h)$ is Euler's totient function and $\nu(h)$ denotes the number of distinct prime factors of h, by the following argument:

Using the first line of the proof of Theorem 3.1 in Harris and Subbarao [2]

$$Q_k(x, h) = \sum_{d<x;(d,h)=1} \mu_k(d) \sum_{n<x/d;(n,h)=1} 1$$

so that

$$Q_k(x; h) = \sum_{d<x;(d,h)=1} \mu_k(d)\left\{\frac{\phi(h)}{h}\frac{x}{d} + O(2^{\nu(h)})\right\}.$$

The leading term of (8) follows from the Harris-Subbarao argument. The error term

$$= O\left(2^{\nu(h)} \sum_{d<x;(d,h)=1} \mu_k^2(d)\right) = O\left(2^{\nu(h)} \sum_{d<x^{1/k};(d,h)=1} 1\right)$$

$$= O\left(2^{\nu(h)}\left\{\frac{\phi(h)}{h} x^{1/k} + O(2^{\nu(h)})\right\}\right) = O\left(2^{\nu(h)} \frac{\phi(h)}{h} x^{1/k}\right) + O(4^{\nu(h)}) .$$

We also remark that if $0 < x \leq y$, then

(9) $$\psi_k(x, y; h) = Q_k(x; h).$$

As for the generalization of the Buchstab identity, we let A denote a nonempty finite set of natural numbers and define

(10) $$G(A, y, g) = \sum_{a\in A;p(a)<y} g(a), \qquad g \text{ multiplicative},$$

then

(11) $$G(A, u, g) - G(A, v, g) = \begin{cases} \sum_{v\leq p<u} \sum_{m=1}^{\infty} g(p^m) G(A_{p^m}, p, g), & 2 \leq v \leq u, \\ 0, & \text{otherwise}, \end{cases}$$

where

(12) $$A_{p^m} = \left\{\frac{a}{p^m} \mid a \in A,\ a \equiv 0 \pmod{p^m},\ a \not\equiv 0 \pmod{p^{m+1}}\right\}.$$

In view of [1] and [3], we will omit the proof of (11).

Now we state the first of 2 lemmas needed to prove the Theorem.

Lemma 1. *If t is a real number such that $1 < t \leq 2$ and h is a natural number, then*

$$\psi_k(y^t, y; h) \geq \prod_{p\mid h}\left(\frac{1-p^{-1}}{1-p^{-k}}\right)\frac{y^t}{\zeta(k)}\left\{1 - \log t + O\left(\frac{4^{\nu(h)} \log\log 3h}{\log y}\right)\right\}.$$

Proof. We take $A = \{n \mid 1 \leq n < y^t,\ (n, h) = 1\}$, $u = y^t$, $v = y$, and $g(n) = f_k(n)$ in (11) so that by (8) and (9), we have

$$\psi_k(y^t, y; h) = \prod_{p\mid h}\left(\frac{1-p^{-1}}{1-p^{-k}}\right)\frac{y^t}{\zeta(k)} + O\left(y^{t/k}\frac{\phi(h)}{h} 2^{\nu(h)}\right)$$

$$+ O(4^{\nu(h)}) - \sum_{y\leq p<y^t;p\nmid h} Q_k\left(\frac{y^t}{p}; h\right),$$

since $y \le p < y^t$ and $1 < t \le 2$ imply $y^t/p \le p$.

Hence

$$\psi_k(y^t, y; h) = \prod_{p|h}\left(\frac{1-p^{-1}}{1-p^{-k}}\right)\frac{y^t}{\zeta(k)}\left\{1 - \sum_{y\le p<y^t; p\nmid h}\frac{1}{p}\right\} + O\left(y^{t/k}\frac{\phi(h)}{h}\,2^{\nu(h)}\right)$$

(14)
$$+ O(4^{\nu(h)}) + O\left(y^{t/k}\frac{\phi(h)}{h}\,2^{\nu(h)}\sum_{y\le p<y^t; p\nmid h} p^{-1/k}\right) + O(4^{\nu(h)}\pi(y^t))$$

$$= \prod_{p|h}\left(\frac{1-p^{-1}}{1-p^{-k}}\right)\frac{y^t}{\zeta(k)}\left\{1 - \sum_{y\le p<y^t; p\nmid h}\frac{1}{p}\right\} + O\left(4^{\nu(h)}\frac{y^t}{\log y}\right).$$

Now, as in [3], we observe that

$$\sum_{p<x; p\nmid h}\frac{1}{p} = \log\log x + \left(B - \sum_{p|h}\frac{1}{p}\right) + O\left(\frac{1}{\log x} + \frac{\nu(h)}{x}\right).$$

Thus with $1 < t \le 2$

$$\psi_k(y^t, y; h) = \prod_{p|h}\left(\frac{1-p^{-1}}{1-p^{-k}}\right)\frac{y^t}{\zeta(k)}\left\{1 - \log t + O\left(\frac{1}{\log y} + \frac{\nu(h)}{y} + \frac{4^{\nu(h)}\log\log 3h}{\log y}\right)\right\}$$

$$= \prod_{p|h}\left(\frac{1-p^{-1}}{1-p^{-k}}\right)\frac{y^t}{\zeta(k)}\left\{1 - \log t + O\left(\frac{4^{\nu(h)}\log\log 3h}{\log y}\right)\right\},$$

to complete the proof of the lemma.

Due to the factor $4^{\nu(h)}$ in the error term, we are forced to restrict h by the condition

(15) $$h \le (\log y)^C$$

for an absolute constant C.

Now we define

(16) $$\Lambda_k(x, y; h) = \left(\prod_{p|h}\left(\frac{1-p^{-1}}{1-p^{-k}}\right)\frac{y^t}{\zeta(k)}\right)^{-1}\psi_k(x, y; h)$$

and state Lemma 2, whose proof is almost identical to that of Lemma 3 of [3] and so will be omitted.

Lemma 2. *Let δ be a number satisfying*

(17) $$0 < \delta < 1/3.$$

For each natural number n, there exists a number $y_1 = y_1(n, \delta)$ such that if $1/3 < t \le n(1-\delta)$, then

(18) $$\Lambda_k(y^t, y; h) \ge (2/n!)\delta^{n-1}$$

for $y \ge y_1$ and $h \le (\log y)^C$ where C is an absolute constant.

The proof of the Theorem follows from Lemma 2 by a suitable choice of

n and δ in terms of t, namely $n = [(t+1)/(1-\delta)]$ and $\delta = 1/(2+\log(t+1))$ (see Halberstam [1]).

REFERENCES

1. H. Halberstam, *On integers all of whose prime factors are small*, Proc. London Math. Soc. (3) **21** (1970), 102–107. MR 42 #4509.

2. V. C. Harris and M. V. Subbarao, *An arithmetic sum with an application to quasi-k-free integers*, J. Austral. Math. Soc. **15** (1973), 272–278. MR **48** #8363.

3. D. G. Hazlewood, *On integers all of whose prime factors are small*, Bull. London Math. Soc. **5** (1973), 159–163.

4. ———, *Sums over positive integers with few prime factors*, J. Number Theory **7** (1975), 189–207.

5. B. V. Levin and A. S. Fainleib, *Application of certain integral equations to questions of the theory of numbers*, Uspehi Mat. Nauk **22** (1967), no. 3 (135), 119–197 = Russian Math. Surveys **22** (1967), no. 3, 119–204. MR 37 #5174.

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