WILDLY RAMIFIED $\mathbb{Z}/2$ ACTIONS IN DIMENSION TWO

M. ARTIN

ABSTRACT. The rings of power series which are invariant under an automorphism of order 2 are described by equations having a standard form.

Let $k$ be a field of characteristic two, and let $k[[u, v]]$ be a power series ring over $k$ in two variables. Our object is to study the ring $R$ of invariants of $k[[u, v]]$ under an involution $\sigma$, i.e., under a $k$-automorphism $\sigma$ of $k[[u, v]]$ of order 2. We assume that the action of $\sigma$ on Spec $k[[u, v]]$ is free except at the closed point. This means that there is no prime ideal $\mathfrak{p}$ other than the maximal ideal which is $\sigma$-invariant, and such that the induced action on $k[[u, v]]/\mathfrak{p}$ is trivial.

It is known that $k[[u, v]]$ is finite over $R$ [1], and in view of our assumption on fixed points, that $k[[u, v]]$ is étale and of degree 2 over $R$ except at the closed point. It follows that $R$ is a complete local ring. Thus we are in effect studying a complete local $k$-algebra $R$ such that the fundamental group of its pointed spectrum $X = \text{Spec } R - \{\mathfrak{m}_R\}$ is $\mathbb{Z}/2$, and that its universal covering is the pointed spectrum $U$ of a regular local $k$-algebra with residue field $k$.

Here is the result:

Theorem. The ring $R$ can be defined in $k[[x, y, z]]$ by one equation of the form

$$z^2 + abz + a^2y + b^2x = 0,$$

where $a, b \in k[[x, y]]$ are nonunits which are relatively prime. Conversely, any such equation defines a ring $R$ having the above properties. Its double cover $k[[u, v]]$ is given by the equations

$$u^2 + au + x = 0, \quad v^2 + bv + y = 0,$$

and if we denote the action of $\sigma$ by a bar, then

$$\bar{u}u = x, \quad \bar{v}v = y, \quad \bar{uv} + \bar{vu} = z.$$ 

It would be interesting to have an extension of this result to $\mathbb{Z}/p$-actions for $p > 2$. 

Received by the editors July 5, 1974.


Key words and phrases. Wild ramification, étale coverings, group actions.

1 Supported by NSF Grant DGP P 12656.
We consider $\sigma$ as a pair of power series in $u$, $v$. The linear terms will be given by a matrix whose square is the identity. After a linear change of variable, the matrix will be of the form \[
abla = \begin{pmatrix} 1 & \gamma \\ 0 & 1 \end{pmatrix}.\] (It turns out that in fact $\gamma = 0$.) This means
\[
\bar{u} = u + (\text{degree} \geq 2), \quad \bar{v} = v + \epsilon u + (\text{degree} \geq 2).
\]

Set
\[
x = uu = u^2 + (\text{degree} \geq 3), \quad y = vv = v^2 + \epsilon uv + (\text{degree} \geq 3).
\]

Then we obviously have $k[[u, v]] \otimes R \supset k[[x, y]]$.

**Lemma 1.** $k[[u, v]]$ and $R$ are free $k[[x, y]]$-algebras, of ranks 4 and 2 respectively.

**Proof.** It is clear that $x$, $y$ form a system of parameters in $k[[u, v]]$ and hence that $k[[u, v]]$ is a finite $k[[x, y]]$-module. It is free by [4, IV-37, Proposition 22]. Thus we need only check that $\dim_k k[[u, v]]/(x, y) = 4$. That is clear—a basis consists of the residues of $1, u, v, uv$. Since $k[[u, v]]$ is generically étale and of degree 2 over $R$, $R$ is of rank 2 over $k[[x, y]]$. Again, it is free by [4, loc. cit.].

**Corollary.** The multiplicity of $R$ is two.

**Lemma 2.** The field extension $k((u, v))$ over $k((x, y))$ is Galois.

**Proof.** Let $K$ be the field of fractions of $R$. Then the field extension $k((u, v))/K$ is separable and unramified in codimension 1 on $R$. Also, $K$ is a separable extension of $k((x, y))$. For, otherwise $R$ would be purely inseparable over $k((x, y))$, and such a ring cannot have any extension unramified in codimension 1 (purity of the branch locus [5], and [2, p. 240, Theorem 4.10]). Since $[K: k((x, y))] = 2$, $K$ is Galois over $k((x, y))$.

Let $S = R \otimes R$ and $T = k[[u, v]] \otimes k[[u, v]]$, both tensor products being over the ring $k[[x, y]]$. Let $\overline{S}$, $\overline{T}$ denote the normalization of these rings. Above any codimension 1 prime of $k[[x, y]]$, the extension $R \to k[[u, v]]$ is étale and of degree 2. Hence $S \to T$ is étale of degree 4 there, and so is $\overline{S} \to \overline{T}$. Since $K$ is Galois, $\overline{S} \approx R \times R$. Therefore $\overline{T}$ is unramified in codimension 1 over $R$ (say with $R$ acting on the left in the tensor product), and so it is certainly unramified over $k[[u, v]]$ in codimension 1. By purity [4], $T$ splits completely as $k[[u, v]]$-algebra. Therefore $k((u, v))$ is Galois.

**Lemma 3.** The Galois group of $k((u, v))/k((x, y))$ is $G = \mathbb{Z}/2 \oplus \mathbb{Z}/2$.

**Proof.** Otherwise, it must be a cyclic group. We know by purity that $R$ is ramified over $k[[x, y]]$ at some codimension 1 prime $\mathfrak{p}$ of $k[[x, y]]$. Let $\mathfrak{q}$ be a prime of $k[[u, v]]$ lying over $\mathfrak{p}$, and let $H \subset G$ be the inertial subgroup
of $\mathcal{Q}$. Then since $\mathcal{Q}$ cannot be ramified over $R$, $H$ has order 2. There is a prime $\mathfrak{P}$ of $k[[u, v]]^H$ which is unramified over $\mathfrak{P}$. If $G$ were cyclic, there would be only one subgroup $H$ of order 2, and so we would have $k[[u, v]]^H = R$. This contradicts the choice of $\mathfrak{P}$.

By Lemma 3, there are exactly two fields $L, L'$ between $k((x, y))$ and $k((u, v))$ besides $K$. Let $A, B$ denote the normalizations of $k[[x, y]]$ in $L$ and $L'$ respectively. These are again free, $k[[x, y]]$-algebras of rank 2. Any such algebra is generated by one element. So we may write

$$A = k[[x, y]][s]/(s^2 + as + \xi), \quad B = k[[x, y]][t]/(t^2 + bt + \eta)$$

with $a, b, \xi, \eta \in k[[x, y]]$. The sets $\{a = 0\}$ and $\{b = 0\}$ are the ramification loci of $A$ and $B$ respectively.

Lemma 4. The elements $a, b$ are relatively prime nonunits in $k[[x, y]]$.

Proof. The $k[[x, y]]$-algebras $A, B$ are ramified, by purity. Hence $a, b$ are not invertible. Let $\mathfrak{P}$ be a codimension 1 prime of $k[[x, y]]$ above which $A$ is ramified. Then $k[[u, v]]$ is also ramified above $\mathfrak{P}$, and hence so is $R$. Let $\mathcal{Q}$ be a prime of $k[[u, v]]$ lying over $\mathfrak{P}$. Then as in the previous lemma, the inertial subgroup $H$ leads to an intermediate ring $k[[u, v]]^H$ which is unramified at some (and hence all) primes over $\mathfrak{P}$. This ring has no choice but to be $B$. Thus $\mathfrak{P}$ does not contain $b$, and so $a$ and $b$ are relatively prime.

Lemma 5. $k[[u, v]] = A \otimes B$, the tensor product being over $k[[x, y]]$.

Proof. Since the ramification loci of $A$ and $B$ have only the closed point in common, $A \otimes B$ is nonsingular in codimension 1. It follows easily that the natural map $A \otimes B \to k[[u, v]]$ is an isomorphism in codimension 1. Both rings are free modules, and so $\phi$ is an isomorphism.

We now view $A \otimes B = k[[u, v]]$ as the ring defined by the equations

$$s^2 + as + \xi = 0, \quad t^2 + bt + \eta = 0$$

in $k[[x, y, s, t]]$, and we apply the jacobian criterion. Since $k[[u, v]]$ is formally smooth over $k$ and $a, b$ are nonunits, it follows that the jacobian matrix

$$\begin{pmatrix}
\frac{\partial \xi}{\partial x} & \frac{\partial \xi}{\partial y} \\
\frac{\partial \eta}{\partial x} & \frac{\partial \eta}{\partial y}
\end{pmatrix}$$

is invertible, hence that $\xi, \eta$ is a regular system of parameters in $k[[x, y]]$. This implies in turn that $s, t$ is a regular system of parameters in $k[[u, v]]$. By construction, the automorphism $\sigma$ is given by the actions on each factor of $A \otimes B$, i.e., we have $\overline{s} = s + a, \overline{t} = t + b$, and $s\overline{t} = \xi, t\overline{s} = \eta$. So, we can make the change of variable $(x, y, u, v) \to (\xi, \eta, s, t)$ to obtain equations

$$(*) \quad u^2 + au + x = 0, \quad v^2 + bv + y = 0.$$
Conversely, let $a, b \in k[[x, y]]$ be any relatively prime nonunits, and consider the extension given by the equations (*). It is immediate by Galois theory that they define a Galois extension with group $G = \mathbb{Z}/2 \oplus \mathbb{Z}/2$. Moreover, the jacobian criterion shows that the ring defined by these equations is smooth, and equal to $k[[u, v]]$. Let $A, B, R$ be the three intermediate rings, where $A = k[[x, y]][u]/(u^2 + au + x)$, and $B = k[[x, y]][v]/(v^2 + bv + y)$.

**Lemma 6.** With the above notation, $k[[u, v]]$ is unramified in codimension 1 over $R$.

**Proof.** Clearly, $k[[u, v]]$ is the normalization of $A \otimes R$. Since $A$ is étale over $k[[x, y]]$ at each point of $U_a = \text{Spec } k[[x, y]][1/a]$, it is clear that $A \otimes R$, and hence $k[[u, v]]$, is étale over $R$ except above the locus $\{a = 0\}$. Similarly, $k[[u, v]]$ is étale over $R$ except above $\{b = 0\}$. Since $a$ and $b$ are relatively prime, the lemma follows.

We now ask for the equation defining $R$. Let $z = uv + vu$, where $\overline{u} = u + a$ and $\overline{v} = v + b$. Clearly $z \in R$, and $z = ub + va$. The irreducible equation for $z$ over $k[[x, y]]$ is easily seen to be

$$f = z^2 + abz + a^2y + b^2x = 0.$$ 

Therefore $k[[x, y, z]]/f$ is birationally equivalent to $R$. It remains to verify that this equation defines a normal ring, i.e., that the ring is nonsingular in codimension 1. This is clear except on the ramification locus $\{ab = 0\}$. Say that $a = 0$, hence $b \neq 0$. At such a point,

$$\frac{\partial f}{\partial x} (\frac{\partial a}{\partial x})bz + b^2.$$ 

Let $a' = \frac{\partial a}{\partial x}$. Then if $\frac{\partial f}{\partial x} = 0$, it follows that $a'z + b = 0$. Substitution of this equality into $f$ leads to $a'^2x = 1$. Since $a'$ is an integral power series but $x$ is not a unit, this cannot hold anywhere on $\text{Spec } k[[x, y, z]]$. This completes the proof of the Theorem.

**Examples.** Let us assume $k$ algebraically closed. Involutions in dimension 1 are easily classified. If $\sigma$ acts on $k[[u]]$, then the invariant ring will be normal, and hence a power series ring $k[[t]]$. By Artin-Schreier theory, we can choose a generator $z$ for the field extension such that

$$z^2 - z = \phi = \sum_{i=-N}^{0} a_i t^i$$

and such that only odd negative indices $a_i$ occur in the expression for $\phi$. Write $\phi = ut^{-2r+1}$, where $u$ is a unit. Then a change of variable $t' = tv$, where $v^{2r-1} = u$ results in an equation

$$z^2 - z = t^{-2r+1}.$$
The element \( s = t'z \) is a local parameter for \( k[[u]] \), satisfying the equation

\[
s^2 + t's + t = 0.
\]

So, this is the normal form in dimension 1. (The only case in which the involution corresponding to this equation is rational is \( r = 1 \), where it is the action

\[
s \rightarrow s/(1 + s) = s + s^2 + s^3 + \cdots.
\]

We can obtain examples in dimension 2 by letting \( \sigma \) act independently on the variables \( u, v \) by some of the above actions. This leads to the cases \( a = x^i, b = y^j \):

\[
u_2 + x^i u + x = 0, \quad v^2 + y^j v + y = 0; \quad z^2 + x^i y^j z + x^2 y + xy^{2j} = 0.
\]

This equation defines a rational singularity [3] if and only if \( i \) or \( j = 1 \). If say \( i = 1 \), it is a double point of type \( D_n \) with \( n = 4j \):

\[
z^2 + xy^j z + x^2 y + xy^{2j} = 0.
\]

Setting \( a = y, b = x^2 \) leads to a rational double point of type \( E_8 \) [3, p. 270]:

\[
z^2 + x^2 yz + y^3 + x^5 = 0.
\]

REFERENCES


