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WILDLY RAMIFIED Z/2 ACTIONS IN DIMENSION TWO

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ABSTRACT. The rings of power series which are invariant under an automorphism of order 2 are described by equations having a standard form.

Let k be a field of characteristic two, and let $k[[u, v]]$ be a power series ring over k in two variables. Our object is to study the ring R of invariants of $k[[u, v]]$ under an involution σ, i.e., under a k-automorphism σ of $k[[u, v]]$ of order 2. We assume that the action of σ on Spec $k[[u, v]]$ is free except at the closed point. This means that there is no prime ideal $\mathfrak{p}$ other than the maximal ideal which is σ-invariant, and such that the induced action on $k[[u, v]]/\mathfrak{p}$ is trivial.

It is known that $k[[u, v]]$ is finite over R [1], and in view of our assumption on fixed points, that $k[[u, v]]$ is étale and of degree 2 over R except at the closed point. It follows that R is a complete local ring. Thus we are in effect studying a complete local k-algebra R such that the fundamental group of its pointed spectrum $X = \operatorname{Spec} R - \{\mathfrak{m}_R\}$ is $\mathbf{Z}/2$, and that its universal covering is the pointed spectrum U of a regular local k-algebra with residue field k.

Here is the result:

Theorem. *The ring R can be defined in $k[[x, y, z]]$ by one equation of the form*

$$z^2 + abz + a^2y + b^2x = 0,$$

where $a, b \in k[[x, y]]$ are nonunits which are relatively prime. Conversely, any such equation defines a ring R having the above properties. Its double cover $k[[u, v]]$ is given by the equations

$$u^2 + au + x = 0, \qquad v^2 + bv + y = 0,$$

and if we denote the action of σ by a bar, then

$$u\bar{u} = x, \qquad v\bar{v} = y, \qquad u\bar{v} + \bar{u}v = z.$$

It would be interesting to have an extension of this result to $\mathbf{Z}/p$-actions for $p > 2$.

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We consider σ as a pair of power series in u, v. The linear terms will be given by a matrix whose square is the identity. After a linear change of variable, the matrix will be of the form $\left\{\begin{smallmatrix}1 & \epsilon\\ 0 & 1\end{smallmatrix}\right\}$. (It turns out that in fact $\epsilon = 0$.) This means

$$\bar{u} = u + (\text{degree} \geq 2), \qquad \bar{v} = v + \epsilon u + (\text{degree} \geq 2).$$

Set

$$x = u\bar{u} = u^2 + (\text{degree} \geq 3), \qquad y = v\bar{v} = v^2 + \epsilon uv + (\text{degree} \geq 3).$$

Then we obviously have $k[[u, v]] \supset R \supset k[[x, y]]$.

Lemma 1. *$k[[u, v]]$ and R are free $k[[x, y]]$-algebras, of ranks 4 and 2 respectively.*

Proof. It is clear that x, y form a system of parameters in $k[[u, v]]$ and hence that $k[[u, v]]$ is a finite $k[[x, y]]$-module. It is free by [4, IV-37, Proposition 22]. Thus we need only check that $\dim_k k[[u, v]]/(x, y) = 4$. That is clear—a basis consists of the residues of $1, u, v, uv$. Since $k[[u, v]]$ is generically étale and of degree 2 over R, R is of rank 2 over $k[[x, y]]$. Again, it is free by [4, loc. cit.].

Corollary. *The multiplicity of R is two.*

Lemma 2. *The field extension $k((u, v))$ over $k((x, y))$ is Galois.*

Proof. Let K be the field of fractions of R. Then the field extension $k((u, v))/K$ is separable and unramified in codimension 1 on R. Also, K is a separable extension of $k((x, y))$. For, otherwise R would be purely inseparable over $k[[x, y]]$, and such a ring cannot have any extension unramified in codimension 1 (purity of the branch locus [5], and [2, p. 240, Theorem 4.10]). Since $[K\colon k((x, y))] = 2$, K is Galois over $k((x, y))$.

Let $S = R \otimes R$ and $T = k[[u, v]] \otimes k[[u, v]]$, both tensor products being over the ring $k[[x, y]]$. Let $\bar{S}$, $\bar{T}$ denote the normalization of these rings. Above any codimension 1 prime of $k[[x, y]]$, the extension $R \to k[[u, v]]$ is étale and of degree 2. Hence $S \to T$ is étale of degree 4 there, and so is $\bar{S} \to \bar{T}$. Since K is Galois, $\bar{S} \approx R \times R$. Therefore $\bar{T}$ is unramified in codimension 1 over R (say with R acting on the left in the tensor product), and so it is certainly unramified over $k[[u, v]]$ in codimension 1. By purity [4], T splits completely as $k[[u, v]]$-algebra. Therefore $k((u, v))$ is Galois.

Lemma 3. *The Galois group of $k((u, v))/k((x, y))$ is $G = \mathbf{Z}/2 \oplus \mathbf{Z}/2$.*

Proof. Otherwise, it must be a cyclic group. We know by purity that R is ramified over $k[[x, y]]$ at some codimension 1 prime $\mathfrak{p}$ of $k[[x, y]]$. Let $\mathfrak{q}$ be a prime of $k[[u, v]]$ lying over $\mathfrak{p}$, and let $H \subset G$ be the inertial subgroup of $\mathfrak{q}$. Then since $\mathfrak{q}$ cannot be ramified over R, H has order 2. There is a prime $\mathfrak{q}_0$ of $k[[u, v]]^H$ which is unramified over $\mathfrak{p}$. If G were cyclic, there would be only one subgroup H of order 2, and so we would have $k[[u, v]]^H = R$. This contradicts the choice of $\mathfrak{p}$.

By Lemma 3, there are exactly two fields L, L' between $k((x, y))$ and $k((u, v))$ besides K. Let A, B denote the normalizations of $k[[x, y]]$ in L and L' respectively. These are again free, $k[[x, y]]$-algebras of rank 2. Any such algebra is generated by one element. So we may write

$$A = k[[x, y]][s]/(s^2 + as + \xi), \qquad B = k[[x, y]][t]/(t^2 + bt + \eta)$$

with $a, b, \xi, \eta \in k[[x, y]]$. The sets $\{a = 0\}$ and $\{b = 0\}$ are the ramification loci of A and B respectively.

Lemma 4. *The elements a, b are relatively prime nonunits in $k[[x, y]]$.*

Proof. The $k[[x, y]]$-algebras A, B are ramified, by purity. Hence a, b are not invertible. Let $\mathfrak{p}$ be a codimension 1 prime of $k[[x, y]]$ above which A is ramified. Then $k[[u, v]]$ is also ramified above $\mathfrak{p}$, and hence so is R. Let $\mathfrak{q}$ be a prime of $k[[u, v]]$ lying over $\mathfrak{p}$. Then as in the previous lemma, the inertial subgroup H leads to an intermediate ring $k[[u, v]]^H$ which is unramified at some (and hence all) primes over $\mathfrak{p}$. This ring has no choice but to be B. Thus $\mathfrak{p}$ does not contain b, and so a and b are relatively prime.

Lemma 5. $k[[u, v]] = A \otimes B$, *the tensor product being over* $k[[x, y]]$.

Proof. Since the ramification loci of A and B have only the closed point in common, $A \otimes B$ is nonsingular in codimension 1. It follows easily that the natural map $A \otimes B \xrightarrow{\phi} k[[u, v]]$ is an isomorphism in codimension 1. Both rings are free modules, and so ϕ is an isomorphism.

We now view $A \otimes B = k[[u, v]]$ as the ring defined by the equations

$$s^2 + as + \xi = 0, \quad t^2 + bt + \eta = 0$$

in $k[[x, y, s, t]]$, and we apply the jacobian criterion. Since $k[[u, v]]$ is formally smooth over k and a, b are nonunits, it follows that the jacobian matrix

$$\begin{pmatrix} \partial\xi/\partial x & \partial\xi/\partial y \\ \partial\eta/\partial x & \partial\eta/\partial y \end{pmatrix}$$

is invertible, hence that ξ, η is a regular system of parameters in $k[[x, y]]$. This implies in turn that s, t is a regular system of parameters in $k[[u, v]]$. By construction, the automorphism σ is given by the actions on each factor of $A \otimes B$, i.e., we have $\bar{s} = s + a$, $\bar{t} = t + b$, and $s\bar{s} = \xi$, $t\bar{t} = \eta$. So, we can make the change of variable $(x, y, u, v) \longrightarrow (\xi, \eta, s, t)$ to obtain equations

$$(*) \qquad u^2 + au + x = 0, \qquad v^2 + bv + y = 0.$$

Conversely, let $a, b \in k[[x, y]]$ be any relatively prime nonunits, and consider the extension given by the equations $(*)$. It is immediate by Galois theory that they define a Galois extension with group $G = \mathbf{Z}/2 \oplus \mathbf{Z}/2$. Moreover, the jacobian criterion shows that the ring defined by these equations is smooth, and equal to $k[[u, v]]$. Let A, B, R be the three intermediate rings, where $A = k[[x, y]][u]/(u^2 + au + x)$, and $B = k[[x, y]][v]/(v^2 + bv + y)$.

Lemma 6. *With the above notation, $k[[u, v]]$ is unramified in codimension 1 over R.*

Proof. Clearly, $k[[u, v]]$ is the normalization of $A \otimes R$. Since A is étale over $k[[x, y]]$ at each point of $U_a = \operatorname{Spec} k[[x, y]][1/a]$, it is clear that $A \otimes R$, and hence $k[[u, v]]$, is étale over R except above the locus $\{a = 0\}$. Similarly, $k[[u, v]]$ is étale over R except above $\{b = 0\}$. Since a and b are relatively prime, the lemma follows.

We now ask for the equation defining R. Let $z = u\bar{v} + \bar{u}v$, where $\bar{u} = u + a$ and $\bar{v} = v + b$. Clearly $z \in R$, and $z = ub + va$. The irreducible equation for z over $k[[x, y]]$ is easily seen to be

$$f = z^2 + abz + a^2y + b^2x = 0.$$

Therefore $k[[x, y, z]]/(f)$ is birationally equivalent to R. It remains to verify that this equation defines a normal ring, i.e., that the ring is nonsingular in codimension 1. This is clear except on the ramification locus $\{ab = 0\}$. Say that $a = 0$, hence $b \neq 0$. At such a point,

$$\partial f/\partial x = (\partial a/\partial x)bz + b^2.$$

Let $a' = \partial a/\partial x$. Then if $\partial f/\partial x = 0$, it follows that $a'z + b = 0$. Substitution of this equality into f leads to $a'^2x = 1$. Since a' is an integral power series but x is not a unit, this cannot hold anywhere on $\operatorname{Spec} k[[x, y, z]]$. This completes the proof of the Theorem.

Examples. Let us assume k algebraically closed. Involutions in dimension 1 are easily classified. If σ acts on $k[[u]]$, then the invariant ring will be normal, and hence a power series ring $k[[t]]$. By Artin-Schreier theory, we can choose a generator z for the field extension such that

$$z^2 - z = \phi = \sum_{i=-N}^{0} a_i t^i$$

and such that only odd negative indices a_i occur in the expression for ϕ. Write $\phi = ut^{-2r+1}$, where u is a unit. Then a change of variable $t' = tv$, where $v^{2r-1} = u$ results in an equation

$$z^2 - z = t^{-2r+1}.$$

The element $s = t^r z$ is a local parameter for $k[[u]]$, satisfying the equation

$$s^2 + t^r s + t = 0.$$

So, this is the normal form in dimension 1. (The only case in which the involution corresponding to this equation is rational is $r = 1$, where it is the action

$$s \rightsquigarrow s/(1+s) = s + s^2 + s^3 + \cdots.)$$

We can obtain examples in dimension 2 by letting σ act independently on the variables u, v by some of the above actions. This leads to the cases $a = x^i$, $b = y^j$:

$$u^2 + x^i u + x = 0, \qquad v^2 + y^j v + y = 0; \qquad z^2 + x^i y^j z + x^{2i} y + x y^{2j} = 0.$$

This equation defines a rational singularity [3] if and only if i or $j = 1$. If say $i = 1$, it is a double point of type D_n with $n = 4j$:

$$z^2 + x y^j z + x^2 y + x y^{2j} = 0.$$

Setting $a = y$, $b = x^2$ leads to a rational double point of type E_8 [3, p. 270]:

$$z^2 + x^2 yz + y^3 + x^5 = 0.$$

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