WILDLY RAMIFIED Z/2 ACTIONS IN DIMENSION TWO

M. ARTIN

ABSTRACT. The rings of power series which are invariant under an automorphism of order 2 are described by equations having a standard form.

Let \( k \) be a field of characteristic two, and let \( k[[u, v]] \) be a power series ring over \( k \) in two variables. Our object is to study the ring \( R \) of invariants of \( k[[u, v]] \) under an involution \( \sigma \), i.e., under a \( k \)-automorphism \( \sigma \) of \( k[[u, v]] \) of order 2. We assume that the action of \( \sigma \) on Spec \( k[[u, v]] \) is free except at the closed point. This means that there is no prime ideal \( \mathfrak{p} \) other than the maximal ideal which is \( \sigma \)-invariant, and such that the induced action on \( k[[u, v]]/\mathfrak{p} \) is trivial.

It is known that \( k[[u, v]] \) is finite over \( R \) [1], and in view of our assumption on fixed points, that \( k[[u, v]] \) is étale and of degree 2 over \( R \) except at the closed point. It follows that \( R \) is a complete local ring. Thus we are in effect studying a complete local \( k \)-algebra \( R \) such that the fundamental group of its pointed spectrum \( X = \text{Spec } R - \{ \mathfrak{m}_R \} \) is \( \mathbb{Z}/2 \), and that its universal covering is the pointed spectrum \( U \) of a regular local \( k \)-algebra with residue field \( k \).

Here is the result:

Theorem. The ring \( R \) can be defined in \( k[[x, y, z]] \) by one equation of the form

\[
z^2 + abz + a^2y + b^2x = 0,
\]

where \( a, b \in k[[x, y]] \) are nonunits which are relatively prime. Conversely, any such equation defines a ring \( R \) having the above properties. Its double cover \( k[[u, v]] \) is given by the equations

\[
u^2 + au + x = 0, \quad v^2 + bv + y = 0,
\]

and if we denote the action of \( \sigma \) by a bar, then

\[
u \bar{u} = x, \quad v \bar{v} = y, \quad u \bar{v} + \bar{u}v = z.
\]

It would be interesting to have an extension of this result to \( \mathbb{Z}/p \)-actions for \( p > 2 \).
We consider $\sigma$ as a pair of power series in $u, v$. The linear terms will be given by a matrix whose square is the identity. After a linear change of variable, the matrix will be of the form $\begin{pmatrix} 1 & \epsilon \\ 0 & 1 \end{pmatrix}$. (It turns out that in fact $\epsilon = 0$.) This means

$$\overline{u} = u + (\text{degree} \geq 2), \quad \overline{v} = v + \epsilon u + (\text{degree} \geq 2).$$

Set

$$x = uu = u^2 + (\text{degree} \geq 3), \quad y = vv = v^2 + \epsilon uv + (\text{degree} \geq 3).$$

Then we obviously have $k[[u, v]] \supset R \supset k[[x, y]]$.

**Lemma 1.** $k[[u, v]]$ and $R$ are free $k[[x, y]]$-algebras, of ranks 4 and 2 respectively.

**Proof.** It is clear that $x, y$ form a system of parameters in $k[[u, v]]$ and hence that $k[[u, v]]$ is a finite $k[[x, y]]$-module. It is free by [4, IV-37, Proposition 22]. Thus we need only check that $\dim_k k[[u, v]]/(x, y) = 4$. That is clear—a basis consists of the residues of $1, u, v, uv$. Since $k[[u, v]]$ is generically étale and of degree 2 over $R$, $R$ is of rank 2 over $k[[x, y]]$. Again, it is free by [4, loc. cit.].

**Corollary.** The multiplicity of $R$ is two.

**Lemma 2.** The field extension $k((u, v))$ over $k((x, y))$ is Galois.

**Proof.** Let $K$ be the field of fractions of $R$. Then the field extension $k((u, v))/K$ is separable and unramified in codimension 1 on $R$. Also, $K$ is a separable extension of $k((x, y))$. For, otherwise $R$ would be purely inseparable over $k[[x, y]]$, and such a ring cannot have any extension unramified in codimension 1 (purity of the branch locus [5], and [2, p. 240, Theorem 4.10]). Since $[K : k((x, y))] = 2$, $K$ is Galois over $k((x, y))$.

Let $S = R \otimes R$ and $T = k[[u, v]] \otimes k[[u, v]]$, both tensor products being over the ring $k[[x, y]]$. Let $\overline{S}, \overline{T}$ denote the normalization of these rings. Above any codimension 1 prime of $k[[x, y]]$, the extension $R \to k[[u, v]]$ is étale and of degree 2. Hence $S \to T$ is étale of degree 4 there, and so is $\overline{S} \to \overline{T}$. Since $K$ is Galois, $\overline{S} \approx R \times R$. Therefore $\overline{T}$ is unramified in codimension 1 over $R$ (say with $R$ acting on the left in the tensor product), and so it is certainly unramified over $k[[u, v]]$ in codimension 1. By purity [4], $T$ splits completely as $k[[u, v]]$-algebra. Therefore $k((u, v))$ is Galois.

**Lemma 3.** The Galois group of $k((u, v))/k((x, y))$ is $G = \mathbb{Z}/2 \oplus \mathbb{Z}/2$.

**Proof.** Otherwise, it must be a cyclic group. We know by purity that $R$ is ramified over $k[[x, y]]$ at some codimension 1 prime $\mathfrak{p}$ of $k[[x, y]]$. Let $\mathfrak{q}$ be a prime of $k[[u, v]]$ lying over $\mathfrak{p}$, and let $H \subset G$ be the inertial subgroup
of \( \mathfrak{q} \). Then since \( \mathfrak{q} \) cannot be ramified over \( R \), \( H \) has order 2. There is a prime \( \mathfrak{q}' \) of \( k[[u, v]]^H \) which is unramified over \( \mathfrak{p} \). If \( G \) were cyclic, there would be only one subgroup \( H \) of order 2, and so we would have \( k[[u, v]]^H = R \). This contradicts the choice of \( \mathfrak{p} \).

By Lemma 3, there are exactly two fields \( L, L' \) between \( k((x, y)) \) and \( k((u, v)) \) besides \( K \). Let \( A, B \) denote the normalizations of \( k[[x, y]] \) in \( L \) and \( L' \) respectively. These are again free, \( k[[x, y]] \)-algebras of rank 2. Any such algebra is generated by one element. So we may write

\[
A = k[[x, y]][s]/(s^2 + as + \xi), \quad B = k[[x, y]][t]/(t^2 + bt + \eta)
\]

with \( a, b, \xi, \eta \in k[[x, y]] \). The sets \( \{a = 0\} \) and \( \{b = 0\} \) are the ramification loci of \( A \) and \( B \) respectively.

**Lemma 4.** The elements \( a, b \) are relatively prime nonunits in \( k[[x, y]] \).

**Proof.** The \( k[[x, y]] \)-algebras \( A, B \) are ramified, by purity. Hence \( a, b \) are not invertible. Let \( \mathfrak{p} \) be a codimension 1 prime of \( k[[x, y]] \) above which \( A \) is ramified. Then \( k[[u, v]] \) is also ramified above \( \mathfrak{p} \) and hence so is \( R \).

Let \( \mathfrak{q} \) be a prime of \( k[[u, v]] \) lying over \( \mathfrak{p} \). Then as in the previous lemma, the inertial subgroup \( H \) leads to an intermediate ring \( k[[u, v]]^H \) which is unramified at some (and hence all) primes over \( \mathfrak{p} \). This ring has no choice but to be \( B \). Thus \( \mathfrak{p} \) does not contain \( b \), and so \( a \) and \( b \) are relatively prime.

**Lemma 5.** \( k[[u, v]] = A \otimes B \), the tensor product being over \( k[[x, y]] \).

**Proof.** Since the ramification loci of \( A \) and \( B \) have only the closed point in common, \( A \otimes B \) is nonsingular in codimension 1. It follows easily that the natural map \( A \otimes B \to k[[u, v]] \) is an isomorphism in codimension 1. Both rings are free modules, and so \( \phi \) is an isomorphism.

We now view \( A \otimes B = k[[u, v]] \) as the ring defined by the equations

\[
s^2 + as + \xi = 0, \quad t^2 + bt + \eta = 0
\]

in \( k[[x, y, s, t]] \), and we apply the jacobian criterion. Since \( k[[u, v]] \) is formally smooth over \( k \) and \( a, b \) are nonunits, it follows that the jacobian matrix

\[
\begin{pmatrix}
\partial \xi/\partial x & \partial \xi/\partial y \\
\partial \eta/\partial x & \partial \eta/\partial y
\end{pmatrix}
\]

is invertible, hence that \( \xi, \eta \) is a regular system of parameters in \( k[[x, y]] \). This implies in turn that \( s, t \) is a regular system of parameters in \( k[[u, v]] \). By construction, the automorphism \( \sigma \) is given by the actions on each factor of \( A \otimes B \), i.e., we have \( \bar{s} = s + a, \bar{t} = t + b \), and \( s\bar{s} = \xi, t\bar{t} = \eta \). So, we can make the change of variable \( (x, y, u, v) \to (\xi, \eta, s, t) \) to obtain equations

\[
(*) \quad u^2 + au + x = 0, \quad v^2 + bv + y = 0.
\]
Conversely, let \( a, b \in k[[x, y]] \) be any relatively prime nonunits, and consider the extension given by the equations \((*)\). It is immediate by Galois theory that they define a Galois extension with group \( G = \mathbb{Z}/2 \oplus \mathbb{Z}/2 \). Moreover, the Jacobian criterion shows that the ring defined by these equations is smooth, and equal to \( k[[u, v]] \). Let \( A, B, R \) be the three intermediate rings, where \( A = k[[x, y]][u]/(u^2 + au + x) \), and \( B = k[[x, y]][v]/(v^2 + bv + y) \).

**Lemma 6.** With the above notation, \( k[[u, v]] \) is unramified in codimension 1 over \( R \).

**Proof.** Clearly, \( k[[u, v]] \) is the normalization of \( A \otimes R \). Since \( A \) is étale over \( k[[x, y]] \) at each point of \( U_a = \text{Spec} \ k[[x, y]][1/a] \), it is clear that \( A \otimes R \), and hence \( k[[u, v]] \), is étale over \( R \) except above the locus \( \{a = 0\} \). Similarly, \( k[[u, v]] \) is étale over \( R \) except above \( \{b = 0\} \). Since \( a \) and \( b \) are relatively prime, the lemma follows.

We now ask for the equation defining \( R \). Let \( z = uv + uv \), where \( u = u + a \) and \( v = v + b \). Clearly \( z \in R \), and \( z = ub + va \). The irreducible equation for \( z \) over \( k[[x, y]] \) is easily seen to be

\[
f = z^2 + abz + a^2y + b^2x = 0.
\]

Therefore \( k[[x, y, z]]/(f) \) is birationally equivalent to \( R \). It remains to verify that this equation defines a normal ring, i.e., that the ring is nonsingular in codimension 1. This is clear except on the ramification locus \( \{ab = 0\} \). Say that \( a = 0 \), hence \( b \neq 0 \). At such a point,

\[
\frac{\partial f}{\partial x} = (\partial a/\partial x)bz + b^2.
\]

Let \( a' = \partial a/\partial x \). Then if \( \frac{\partial f}{\partial x} = 0 \), it follows that \( a'z + b = 0 \). Substitution of this equality into \( f \) leads to \( a'^2x = 1 \). Since \( a' \) is an integral power series but \( x \) is not a unit, this cannot hold anywhere on \( \text{Spec} \ k[[x, y, z]] \). This completes the proof of the Theorem.

**Examples.** Let us assume \( k \) algebraically closed. Involutions in dimension 1 are easily classified. If \( \sigma \) acts on \( k[[u]] \), then the invariant ring will be normal, and hence a power series ring \( k[[t]] \). By Artin-Schreier theory, we can choose a generator \( z \) for the field extension such that

\[
z^2 - z = \phi = \sum_{i=0}^{2r+1} a_i t^i
\]

and such that only odd negative indices \( a_i \) occur in the expression for \( \phi \). Write \( \phi = ut^{-2r+1} \), where \( u \) is a unit. Then a change of variable \( t' = tv \), where \( v^{2r-1} = u \) results in an equation

\[
z^2 - z = t^{-2r+1}.
\]

License or copyright restrictions may apply to redistribution; see https://www.ams.org/journal-terms-of-use
The element \( s = t'Tz \) is a local parameter for \( k[[u]] \), satisfying the equation

\[
s^2 + ts + t = 0.
\]

So, this is the normal form in dimension 1. (The only case in which the involution corresponding to this equation is rational is \( r = 1 \), where it is the action

\[
s \mapsto s/(1 + s) = s + s^2 + s^3 + \cdots.
\]

We can obtain examples in dimension 2 by letting \( \sigma \) act independently on the variables \( u, v \) by some of the above actions. This leads to the cases \( a = x^i, b = y^j \):

\[
\begin{align*}
u^2 + x'u + x &= 0, \\
v^2 + y'v + y &= 0; \\
z^2 + x'y'z + x^2y + xy^2z &= 0.
\end{align*}
\]

This equation defines a rational singularity [3] if and only if \( i \) or \( j = 1 \). If say \( i = 1 \), it is a double point of type \( D_n \) with \( n = 4j \):

\[
z^2 + xy'z + x^2y + xy^2z = 0.
\]

Setting \( a = y, b = x \) leads to a rational double point of type \( E_6 \) [3, p. 270]:

\[
z^2 + x^3yz + y^3 + x^5 = 0.
\]

REFERENCES


