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ORDER IN A SPECIAL CLASS OF RINGS AND A STRUCTURE THEOREM

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ABSTRACT. Below a special class of not necessarily associative or commutative rings A is considered which is characterized by the property that A has no nonzero nilpotent element and that a product of elements of A which is equal to zero remains equal to zero no matter how its factors are associated. It is shown that $(A, \leq)$ is a partially ordered set where $x \leq y$ if and only if $xy = x^2$. Also it is shown that $(A, \leq)$ is infinitely distributive, i.e., $r \sup x_i = \sup rx_i$. Finally, based on Zorn's lemma it is shown that A is isomorphic to a subdirect product of not necessarily associative or commutative rings without zero divisors.

In what follows A stands for a not necessarily associative or commutative ring satisfying property (a) given by:

(a) *A has no nilpotent element of index 2, and a product of elements of A which is equal to zero remains equal to zero no matter how its factors are associated.*

For the sake of brevity, the second property of A mentioned in (a) is rephrased as "*A is associative for products equal to zero*".

Let us observe immediately that A has no nonzero nilpotent element. Indeed, let $x^n = 0$ (a notation which is justified in view of (a)) and $x^{n-1} \neq 0$ for some $n > 2$. Then $x^{n+n-2} = 0 = (x^{n-1})^2$ which by (a) implies $x^{n-1} = 0$, contradicting $x^{n-1} \neq 0$. Thus, (a) is equivalent to

(a_1) *A has no nonzero nilpotent element and A is associative for products equal to zero.*

Let x and y be elements of A. If $xy = 0$ then $y((xy)x) = 0$ which by (a) implies $(yx)(yx) = 0 = (yx)^2$ which, again by (a), implies $yx = 0$. Thus, for every element x and y of A we have $xy = 0$ *implies* $yx = 0$.

Let x, y, z be elements of A. If $xy = 0$ then from the above it follows that $z(yx) = (zy)x = xzy = 0$. Thus, for every element x, y, z of A we have $xy = 0$ *implies* $xzy = 0$.

Let $x_1 \cdots x_m$ be a product of (not necessarily distinct) elements x_i of A and let $y_1 \cdots y_n$ be a product (in any order whatsoever) of all the distinct factors appearing in $x_1 \cdots x_m$. If $x_1 \cdots x_m = 0$ then from the above two implications it follows that $(y_1 \cdots y_n)^m = 0$ which by (a_1) implies $y_1 \cdots$

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$y_n = 0$. Conversely, if $y_1 \cdots y_n = 0$ then from the above two implications it follows that $x_1 \cdots x_m = 0$. But then clearly, (a_1) is equivalent to

(a_2) *A product p of elements of A is equal to zero if and only if every product in which all the distinct factors of p appear is equal to zero.*

It is also easy to verify that (a_2) is equivalent to

(a_3) *A has no nonzero nilpotent element and A is associative and commutative for products equal to zero.*

From the above we see that any one of properties (a) to (a_3) can be used to define A.

Theorem 1. *The ring A is partially ordered by $\le$ where for every element x and y of A*

$$x \le y \text{ if and only if } xy = x^2. \tag{1}$$

Proof. Since $xx = x^2$ it follows from (1) that $x \le x$. Thus, $\le$ is reflexive.

Moreover, if $x \le y$ and $y \le x$ then by (1) we have $xy = x^2$ and $yx = y^2$ so that $x^2 - xy - yx + y^2 = (x - y)^2 = 0$. But then from (a) it follows that $x - y = 0$ and hence $x = y$. Thus, $\le$ is antisymmetric.

Furthermore, let $x \le y$ and $y \le z$. Then by (1) we have

$$xy = x^2 \quad and \quad yz = y^2, \quad i.e., \ y(z - y) = 0. \tag{2}$$

But then by (a_2) and (2) we have

$$0 = xy(z - y) = x^2(z - y) = x(xz - xy) = x(xz - x^2) = x^2(z - x)$$

which by (a_2) implies $x(z - x) = 0$ which, in turn, by (1) implies $x \le z$. Thus, $\le$ is transitive. Hence, $(A, \le)$ is a partially ordered set.

Lemma 1. *For every element x, y, u, v of A*

$$x \le y \text{ and } u \le v \text{ imply } xu \le yv. \tag{3}$$

Proof. From the hypotheses of (3) and (1) it follows that

$$x(y - x) = u(v - u) = 0. \tag{4}$$

But then by (a_2) we have

$$0 = xu(y - x)v = (xu)(yv) - (xu)(xv). \tag{5}$$

Again, by (4) and (a_2) we have

$$0 = xux(v - u) = (xu)(xv) - (xu)(xu)$$

which, in view of (5), implies $(xu)(yv) - (xu)(xu) = 0$ which, in turn, by (1), implies $xu \le yv$, as desired.

The following theorem shows that the infinite distributivity (which is not valid in general in every partially ordered set) is valid in $(A, \leq)$.

Theorem 2. *Let $(x_i)_{i\in I}$ be a subset of A such that $\sup_i x_i$ exists. Then for every element r of A, $\sup_i rx_i$ exists and*

(6) $$r \sup_i x_i = \sup_i rx_i \quad \text{with } i \in I.$$

Proof. For the sake of simplicity we denote $\sup_i x_i$ by $\sup x_i$. Since $x_i \leq \sup x_i$, from (3) it follows that

(7) $$rx_i \leq r \sup x_i \quad \text{with } i \in I.$$

Therefore, $r \sup x_i$ is an upper bound of the set $(rx_i)_{i\in I}$. Let u be any upper bound of the set $(rx_i)_{i\in I}$. Then by (1) we have

(8) $$(rx_i)u = (rx_i)(rx_i) \quad \text{with } i \in I.$$

On the other hand, (7), in view of (1), implies

$$(rx_i)(r \sup x_i) = (rx_i)(rx_i)$$

which, by (8) implies $(rx_i)u = (rx_i)(r \sup x_i)$. Consequently, $(rx_i)(u - r \sup x_i) = 0$, which, in view of (a_2) implies

(9) $$x_i r(u - r \sup x_i) = 0 \quad \text{with } i \in I.$$

Since $x_i \leq \sup x_i$, by (1) we have $x_i \sup x_i = x_i^2$ which by (9) yields

$$x_i(r(u - r \sup x_i) + \sup x_i) = x_i^2,$$

from which, in view of (1), we obtain

$$x_i \leq r(u - r \sup x_i) + \sup x_i \quad \text{with } i \in I.$$

But then, since the right side of the above inequality does not depend on i, we have $\sup x_i \leq r(u - r \sup x_i) + \sup x_i$ so that by (1) we derive

$$\sup x_i (r(u - r \sup x_i) + \sup x_i) = (\sup x_i)(\sup x_i)$$

which implies $\sup x_i(r(u - r \sup x_i)) = 0$, so that by (a_2) we obtain $(r \sup x_i)(u - r \sup x_i) = 0$ or $(r \sup x_i)u = (r \sup x_i)(r \sup x_i)$ and consequently, in view of (1), we have

(10) $$r \sup x_i \leq u.$$

Since, as mentioned above, $r \sup x_i$ is an upper bound of $(rx_i)_{i\in I}$ and u is any upper bound of $(rx_i)_{i\in I}$, it follows that (10) implies (6), as desired.

Remark. With a proof similar to the above it can be shown that if $\sup x_i$ exists then $\sup x_i r$ exists and

(11) $$(\sup x_i)r = \sup x_i r.$$

We observe also that Theorems 1, 2 and (11) are proved without the use of the axiom of choice (or Zorn's lemma).

As usual, a subset H of A is called a *multiplicative system* if and only if H is closed under multiplication, i.e., $x \in H$ and $y \in H$ imply $xy \in H$.

From Zorn's lemma it follows readily that every multiplicative system not containing 0 is a subset of a multiplicative system maximal with respect to the property of not containing 0. Thus, if M is such a maximal multiplicative system then for every $x \in (A - M)$ the smallest (w.r.t. $\subseteq$) multiplicative system $M(x)$ containing M (as a subset) and x (as an element) is such that

(12) $$0 \in M(x).$$

Since A has no nonzero nilpotent element, we see that if h is a nonzero element of A then the set of all the finite products whose factors consist solely of h is a multiplicative system containing h and not containing 0. Thus, from (a) and Zorn's lemma, we have

(13) *Every nonzero element of A is contained in a multiplicative system maximal with respect to the property of not containing* 0.

As usual, an ideal P of A is called a *completely prime* ideal of A if and only if $xy \in P$ implies $x \in P$ or $y \in P$ for every element x and y of A (i.e., if and only if A/P has no zero divisors).

Lemma 2. *Let M be a multiplicative system maximal with respect to the property of not containing* 0. *Then $A - M$ is a completely prime ideal of A.*

Proof. First we show that $A - M$ is closed under subtraction. Assume on the contrary that for some elements p and q of A it is the case that

(14) $$p \in (A - M) \quad \textit{and} \quad q \in (A - M) \quad \textit{and} \quad (p - q) \in M.$$

From (12) and (14) we see that the smallest multiplicative system $M(p)$ containing M (as a subset) and p (as an element) is such that $0 \in M(p)$. Thus, 0 is equal to a product whose factors consist solely of elements of M and p. However, since M is a multiplicative system, by (a_2), we have

(15) $$0 = m_1 p \quad \textit{with} \quad m_1 \in M.$$

Similarly, from (12) and (14) it follows that $0 \in M(q)$, and, as in the above, we have

(16) $$0 = m_2 q \quad \textit{with} \quad m_2 \in M.$$

But then, from (15), (16) and (a_2) we have

$$m_1 m_2 p = m_1 m_2 q = m_1 m_2 (p - q) = 0$$

which, in view of (14) and the fact that $m_1 m_2 \in M$ implies $0 \in M$, contradicting $0 \notin M$. Thus, our assumption is false and $A - M$ is closed under subtraction.

Next, we show that $A - M$ is closed under (left and right) multiplication by elements of A. Assume on the contrary that for some elements p and r of A it is the case that

$$(17) \qquad p \in (A - M) \quad \text{and} \quad pr \in M \quad (\text{or } rp \in M).$$

Let $M(p)$ be the smallest multiplicative system as described above. But then again we see that (17) implies (15), which, in turn, by (a_2) implies $m_1 pr = 0 = m_1 rp$ (with $m_1 \in M$). Hence, from (17), it follows (under either assumption) that $0 \in M$, contradicting $0 \notin M$. Thus, our assumption is false and $A - M$ is closed under (left and right) multiplication by elements of A.

From the above it follows that $A - M$ is an ideal of A. Moreover, $A - M$ is a completely prime ideal of A since M is a multiplicative system.

Theorem 3. *The ring A is isomorphic to a subdirect product of (not necessarily associative or commutative) rings without zero divisors.*

Proof. From (13) and the lemma, it follows that for every nonzero element h of A there exists a completely prime ideal P_i of A such that $h \notin P_i$. Thus, the intersection of all the completely prime ideals of A is $\{0\}$. But then it is well known that A is isomorphic to the subdirect product of the quotient rings A/P_i where P_i ranges over all the completely prime ideals of A. As mentioned previously, A/P_i is a ring without zero divisors. Thus, the theorem is proved.

The results of this paper are applicable to a variety of special cases of rings which satisfy property (a). For these and related applications see the references below.

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