ORDER IN A SPECIAL CLASS OF RINGS
AND A STRUCTURE THEOREM
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ABSTRACT. Below a special class of not necessarily associative or commutative rings $A$ is considered which is characterized by the property that $A$ has no nonzero nilpotent element and that a product of elements of $A$ which is equal to zero remains equal to zero no matter how its factors are associated. It is shown that $(A, \leq)$ is a partially ordered set where $x < y$ if and only if $xy = x^2$. Also it is shown that $(A, \leq)$ is infinitely distributive, i.e., $r \sup_{x_i} x = \sup_{r \sup_{x_i} x_i}$. Finally, based on Zorn's lemma it is shown that $A$ is isomorphic to a subdirect product of not necessarily associative or commutative rings without zero divisors.

In what follows $A$ stands for a not necessarily associative or commutative ring satisfying property (a) given by:

(a) $A$ has no nilpotent element of index 2, and a product of elements of $A$ which is equal to zero remains equal to zero no matter how its factors are associated.

For the sake of brevity, the second property of $A$ mentioned in (a) is rephrased as "$A$ is associative for products equal to zero".

Let us observe immediately that $A$ has no nonzero nilpotent element. Indeed, let $x^n = 0$ (a notation which is justified in view of (a)) and $x^{n-1} \neq 0$ for some $n > 2$. Then $x^{n+n-2} = 0 = (x^{n-1})^2$ which by (a) implies $x^{n-1} = 0$, contradicting $x^{n-1} \neq 0$. Thus, (a) is equivalent to

(a$_1$) $A$ has no nonzero nilpotent element and $A$ is associative for products equal to zero.

Let $x$ and $y$ be elements of $A$. If $xy = 0$ then $y((xy)x) = 0$ which by (a) implies $(yx)(yx) = 0 = (yx)^2$ which, again by (a), implies $yx = 0$. Thus, for every element $x$ and $y$ of $A$ we have $xy = 0$ implies $yx = 0$.

Let $x, y, z$ be elements of $A$. If $xy = 0$ then from the above it follows that $z(yx) = (zy)x = xzy = 0$. Thus, for every element $x, y, z$ of $A$ we have $xy = 0$ implies $xzy = 0$.

Let $x_1 \cdots x_m$ be a product of (not necessarily distinct) elements $x_i$ of $A$ and let $y_1 \cdots y_n$ be a product (in any order whatsoever) of all the distinct factors appearing in $x_1 \cdots x_m$. If $x_1 \cdots x_m = 0$ then from the above two implications it follows that $(y_1 \cdots y_n)^m = 0$ which by (a$_1$) implies $y_1 \cdots y_n = 0$. 

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$y_n = 0$. Conversely, if $y_1 \cdots y_n = 0$ then from the above two implications it follows that $x_1 \cdots x_m = 0$. But then clearly, $(a_1)$ is equivalent to

\( A \) product $p$ of elements of $A$ is equal to zero if and only if every product in which all the distinct factors of $p$ appear is equal to zero.

It is also easy to verify that $(a_2)$ is equivalent to

\( A \) has no nonzero nilpotent element and $A$ is associative and commutative for products equal to zero.

From the above we see that any one of properties $(a)$ to $(a_3)$ can be used to define $A$.

**Theorem 1.** The ring $A$ is partially ordered by $\leq$ where for every element $x$ and $y$ of $A$

\[ x \leq y \text{ if and only if } xy = x^2. \]

**Proof.** Since $xx = x^2$ it follows from (1) that $x \leq x$. Thus, $\leq$ is reflexive.

Moreover, if $x \leq y$ and $y \leq x$ then by (1) we have $xy = x^2$ and $yx = y^2$ so that $x^2 - xy - yx + y^2 = (x - y)^2 = 0$. But then from (a) it follows that $x - y = 0$ and hence $x = y$. Thus, $\leq$ is antisymmetric.

Furthermore, let $x \leq y$ and $y \leq z$. Then by (1) we have

\[ xy = x^2 \text{ and } yz = y^2, \text{ i.e., } y(z - y) = 0. \]

But then by $(a_2)$ and (2) we have

\[ 0 = xy(z - y) = x^2(z - y) = x(zx - xy) = x(xz - x^2) = x^2(z - x) \]

which by $(a_2)$ implies $x(z - x) = 0$ which, in turn, by (1) implies $x \leq z$. Thus, $\leq$ is transitive. Hence, $(A, \leq)$ is a partially ordered set.

**Lemma 1.** For every element $x, y, u, v$ of $A$

\[ x \leq y \text{ and } u \leq v \implies xu \leq yv. \]

**Proof.** From the hypotheses of (3) and (1) it follows that

\[ x(y - x) = u(v - u) = 0. \]

But then by $(a_2)$ we have

\[ 0 = xu(y - x)v = (xu)(yv) - (xu)(xv). \]

Again, by (4) and $(a_2)$ we have

\[ 0 = xu(x - u) = (xu)(xu) - (xu)(xu) \]

which, in view of (5), implies $(xu)(yv) - (xu)(xu) = 0$ which, in turn, by (1), implies $xu \leq yv$, as desired.
The following theorem shows that the infinite distributivity (which is not valid in general in every partially ordered set) is valid in \((A, \leq)\).

**Theorem 2.** Let \((x_i)_{i \in I}\) be a subset of \(A\) such that \(\text{sup}_{i \in I} x_i\) exists. Then for every element \(r\) of \(A\), \(\text{sup}_{i \in I} rx_i\) exists and

\[
(6) \quad r \text{ sup } x_i = \text{ sup } rx_i \quad \text{with } i \in I.
\]

**Proof.** For the sake of simplicity we denote \(\text{sup}_{i \in I} x_i\) by \(\text{sup } x_i\). Since \(x_i \leq \text{sup } x_i\), from (3) it follows that

\[
(7) \quad rx_i \leq r \text{ sup } x_i \quad \text{with } i \in I.
\]

Therefore, \(r \text{ sup } x_i\) is an upper bound of the set \((rx_i)_{i \in I}\). Let \(u\) be any upper bound of the set \((rx_i)_{i \in I}\). Then by (1) we have

\[
(8) \quad (rx_i)u = (rx_i)(rx_i) \quad \text{with } i \in I.
\]

On the other hand, (7), in view of (1), implies

\[
(rx_i)(r \text{ sup } x_i) = (rx_i)(rx_i)
\]

which, by (8) implies \((rx_i)u = (rx_i)(r \text{ sup } x_i)\). Consequently, \((rx_i)(u - r \text{ sup } x_i) = 0\), which, in view of \((a_2)\) implies

\[
(9) \quad x_i(u - r \text{ sup } x_i) = 0 \quad \text{with } i \in I.
\]

Since \(x_i \leq \text{sup } x_i\), by (1) we have \(x_i \text{ sup } x_i = x_i^2\) which by (9) yields

\[
x_i(r(u - r \text{ sup } x_i) + \text{ sup } x_i) = x_i^2,
\]

from which, in view of (1), we obtain

\[
x_i \leq r(u - r \text{ sup } x_i) + \text{ sup } x_i \quad \text{with } i \in I.
\]

But then, since the right side of the above inequality does not depend on \(i\), we have \(\text{sup } x_i \leq r(u - r \text{ sup } x_i) + \text{ sup } x_i\) so that by (1) we derive

\[
\text{sup } x_i (r(u - r \text{ sup } x_i) + \text{ sup } x_i) = (\text{sup } x_i)(\text{sup } x_i)
\]

which implies \(\text{sup } x_i(r(u - r \text{ sup } x_i)) = 0\), so that by \((a_2)\) we obtain

\[
(r \text{ sup } x_i)(u - r \text{ sup } x_i) = 0 \quad \text{or } (r \text{ sup } x_i)u = (r \text{ sup } x_i)(r \text{ sup } x_i)\]

and consequently, in view of (1), we have

\[
(10) \quad r \text{ sup } x_i \leq u.
\]

Since, as mentioned above, \(r \text{ sup } x_i\) is an upper bound of \((rx_i)_{i \in I}\) and \(u\) is any upper bound of \((rx_i)_{i \in I}\), it follows that (10) implies (6), as desired.
Remark. With a proof similar to the above it can be shown that if $\sup x_i$ exists then $\sup x_i r$ exists and

$$\text{(11)} \quad (\sup x_i) r = \sup x_i r.$$ 

We observe also that Theorems 1, 2 and (11) are proved without the use of the axiom of choice (or Zorn's lemma).

As usual, a subset $H$ of $A$ is called a multiplicative system if and only if $H$ is closed under multiplication, i.e., $x \in H$ and $y \in H$ imply $xy \in H$.

From Zorn's lemma it follows readily that every multiplicative system not containing 0 is a subset of a multiplicative system maximal with respect to the property of not containing 0. Thus, if $M$ is such a maximal multiplicative system then for every $x \in (A - M)$ the smallest (w.r.t. $\subseteq$) multiplicative system $M(x)$ containing $M$ (as a subset) and $x$ (as an element) is such that

$$\text{(12)} \quad 0 \in M(x).$$

Since $A$ has no nonzero nilpotent element, we see that if $h$ is a nonzero element of $A$ then the set of all the finite products whose factors consist solely of $h$ is a multiplicative system containing $h$ and not containing 0. Thus, from (a) and Zorn's lemma, we have

$$\text{(13)} \quad \text{Every nonzero element of } A \text{ is contained in a multiplicative system maximal with respect to the property of not containing 0.}$$

As usual, an ideal $P$ of $A$ is called a completely prime ideal of $A$ if and only if $xy \in P$ implies $x \in P$ or $y \in P$ for every element $x$ and $y$ of $A$ (i.e., if and only if $A/P$ has no zero divisors).

Lemma 2. Let $M$ be a multiplicative system maximal with respect to the property of not containing 0. Then $A - M$ is a completely prime ideal of $A$.

Proof. First we show that $A - M$ is closed under subtraction. Assume on the contrary that for some elements $p$ and $q$ of $A$ it is the case that

$$\text{(14)} \quad p \in (A - M) \quad \text{and} \quad q \in (A - M) \quad \text{and} \quad (p - q) \in M.$$ 

From (12) and (14) we see that the smallest multiplicative system $M(p)$ containing $M$ (as a subset) and $p$ (as an element) is such that $0 \in M(p)$.

Thus, 0 is equal to a product whose factors consist solely of elements of $M$ and $p$. However, since $M$ is a multiplicative system, by (a) we have

$$\text{(15)} \quad 0 = m_1 p \quad \text{with} \quad m_1 \in M.$$ 

Similarly, from (12) and (14) it follows that $0 \in M(q)$, and, as in the above, we have

$$\text{(16)} \quad 0 = m_2 q \quad \text{with} \quad m_2 \in M.$$
But then, from (15), (16) and (a2) we have
\[ m_1m_2p = m_1m_2q = m_1m_2(p - q) = 0 \]
which, in view of (14) and the fact that \( m_1m_2 \in M \) implies \( 0 \in M \), contradicting \( 0 \notin M \). Thus, our assumption is false and \( A - M \) is closed under subtraction.

Next, we show that \( A - M \) is closed under (left and right) multiplication by elements of \( A \). Assume on the contrary that for some elements \( p \) and \( r \) of \( A \) it is the case that
\[ p \in (A - M) \quad \text{and} \quad pr \in M \quad \text{(or} \quad rp \notin M). \]

Let \( M(p) \) be the smallest multiplicative system as described above. But then again we see that (17) implies (15), which, in turn, by (a2) implies \( m_1pr = 0 = m_1rp \) (with \( m_1 \in M \)). Hence, from (17), it follows (under either assumption) that \( 0 \in M \), contradicting \( 0 \notin M \). Thus, our assumption is false and \( A - M \) is closed under (left and right) multiplication by elements of \( A \).

From the above it follows that \( A - M \) is an ideal of \( A \). Moreover, \( A - M \) is a completely prime ideal of \( A \) since \( M \) is a multiplicative system.

**Theorem 3.** The ring \( A \) is isomorphic to a subdirect product of (not necessarily associative or commutative) rings without zero divisors.

**Proof.** From (13) and the lemma, it follows that for every nonzero element \( h \) of \( A \) there exists a completely prime ideal \( P_i \) of \( A \) such that \( h \notin P_i \). Thus, the intersection of all the completely prime ideals of \( A \) is \( \{0\} \). But then it is well known that \( A \) is isomorphic to the subdirect product of the quotient rings \( A/P_i \) where \( P_i \) ranges over all the completely prime ideals of \( A \). As mentioned previously, \( A/P_i \) is a ring without zero divisors. Thus, the theorem is proved.

The results of this paper are applicable to a variety of special cases of rings which satisfy property (a). For these and related applications see the references below.

**REFERENCES**