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ON THE SOLVABILITY OF GROUPS OF CENTRAL TYPE[1]

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ABSTRACT. Let G be a finite group with center Z and irreducible complex character χ so that $\chi(1)^2 = [G:Z]$. If the 2-Sylow subgroup of G/Z has order 16 or less then G is solvable.

Introduction. Let G be a finite group with center Z. If G has an irreducible complex character χ with $\chi(1)^2 = [G:Z]$, then G is called a group of central type [4]. It was conjectured in [8] that groups of central type are solvable and several authors have given partial results in this direction [2]–[4], [6], [9], and [12]. For example, in [4] it is proved that if G is a group of central type and if for any prime p, $p^m | [G:Z]$ implies $m \leq 2$ then G is solvable. In [6], the integer 2 in this result was replaced by 4. Here we show that if $2^m | [G:Z]$ implies $m \leq 4$ and G is of central type, then G is solvable. The proof employs the well-known characterizations of simple groups with small 2-Sylow subgroups and information on possible homomorphic images of groups of central type given in [6]. All unexplained notation and terminology is as in [7].

Lemma 1. *Let G be a finite group having a nonabelian composition factor S appearing exactly n times. Then there exist a homomorphic image X of G and an integer m such that $1 \leq m \leq n$ and $S_1 \times \ldots \times S_m \leq X \leq \mathrm{Aut}(S_1 \times \ldots \times S_m)$ where $S_i \simeq S$ for all $i = 1, 2, \ldots, m$.*

Proof. Use induction on $|G|$. Let T be a minimal nontrivial normal subgroup of G. If G is simple then $G = S$ and $S \leq G \leq \mathrm{Aut}(S)$. So we may assume $\{1\} < T < G$. We have $T = T_1 \times \ldots \times T_k$ where the T_i are isomorphic simple groups, $i = 1, 2, \ldots, k$.

Case 1. If $T_i \not\simeq S$ for all $i = 1, 2, \ldots, k$, then consider $G_1 = G/T$. By the inductive hypothesis there exists a homomorphic image X of G_1 and an integer m such that $1 \leq m \leq n$ and $S_1 \times \ldots \times S_m \leq X \leq \mathrm{Aut}(S_1 \times \ldots \times S_m)$, where $S_i \simeq S$. This is the desired homomorphic image X of G.

Case 2. If $T_i \simeq S$ for all $i = 1, 2, \ldots, k$, then $T \simeq S_1 \times \ldots \times S_k$, $k \leq n$ and $S_i \simeq S$. Since $T \triangle G$, $C_G(T) \triangle G$ and we have $G/C_G(T) \leq \mathrm{Aut}(T)$. Since T is minimal normal, $C_G(T) \cap T = \{1\}$ and thus

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$$T \simeq T \cdot C_G(T)/C_G(T) \leq G/C_G(T) \leq \mathrm{Aut}(T).$$

Lemma 2. *Let G be a group of central type with a normal subgroup K such that G/K has a cyclic p-Sylow subgroup. Then G/K has a p-complement. Furthermore, if G/K is also simple then the prime p is unique.*

Proof. [6, 2.4(a) and its proof].

Lemma 3. *Suppose $S \leq X \leq \mathrm{Aut}(S)$, where $S = PSL(2, q)$, q odd and $q > 3$, or $S = A_7$. Then X is not the homomorphic image of a group of central type.*

Proof. [6, 4.3 and the proof of 4.4]..

The following number theoretic results are needed.

Lemma 4 (Zigmundy, 1896). *Let a and n be integers both greater than one. Then there exists a prime divisor p of $a^n - 1$ such that $p \nmid a^m - 1$ for all m satisfying $1 \leq m < n$, except for the following cases:*

(a) *$n = 2$ and a is a Mersenne number.*

(b) *$n = 6$ and $a = 2$.*

Proof. [1].

Lemma 5. *Let a and k be positive integers. Then there exists an odd prime p such that $p \mid a^{2k+1} - 1$ and $p \nmid 2k + 1$.*

Proof. By Lemma 4, there is a prime divisor p of $a^{2k+1} - 1$ such that $p \nmid a^m - 1$ for all $1 \leq m < 2k + 1$. $p \mid a^{p-1} - 1$ by Fermat's little theorem. Hence $p - 1 \geq 2k + 1$, so $p > 2k + 1$. Therefore p is odd and $p \nmid 2k + 1$.

Theorem 1. *Let G be a group of central type with center Z. If the 2-Sylow subgroup of G/Z has order 16 or less, then G is solvable.*

Proof. If the 2-Sylow subgroup of G/Z is trivial then G/Z has odd order and hence is solvable. Since $[G:Z]$ is a square, the 2-Sylow subgroup of G/Z has order 4 or 16. If G/Z is nonsolvable, then it must have at least one nonabelian composition factor. There are several possibilities and these possibilities are considered by cases.

Case 1. Assume a 2-Sylow subgroup P of G/Z has order 4. Then $P = C_2 \times C_2$ by [4, Theorem 2 and Lemma 2]. If G/Z has a nonabelian composition factor S then its 2-Sylow subgroup must be $C_2 \times C_2$, since no nonabelian simple group can have C_2 as its 2-Sylow subgroup. Also S appears exactly once since the 2-Sylow subgroup of G/Z has order 4. Applying Lemma 1, $S \leq X \leq \mathrm{Aut}(S)$ where X is a homomorphic image of G/Z. Since S is a simple group whose 2-Sylow subgroup is $C_2 \times C_2$, $S = PSL(2, p^n)$ where $p^n > 3$ and p is an odd prime [5, 4.2.3]. So $PSL(2, p^n) \leq X \leq \mathrm{Aut}(PSL(2, p^n))$. By Lemma 3, X is not a homomorphic image of a group of central type and this contradiction completes Case 1.

Case 2. Assume a 2-Sylow subgroup of G/Z has order 16.

Subcase 1. Assume G/Z has a nonabelian composition factor S having a 2-Sylow subgroup P of order 16. The only groups of order 16 which can occur as 2-Sylow subgroups of simple groups are the elementary abelian, the dihedral, and the semidihedral groups [5]. P is of central type [4, Theorem 2] and so by [6, 3.2], the dihedral and semidihedral groups are eliminated since they have cyclic self-centralizing subgroups. Hence P is elementary abelian of order 16. By [5, 4.2.3], $S = PSL(2, q)$, $q \equiv \pm 3 \pmod 8$ or $S = SL_2(16)$. The first possibility has already been ruled out in Case 1, so suppose $S = SL_2(16)$. By Lemma 1,

$$SL_2(16) \leq X \leq \mathrm{Aut}(SL_2(16)) = \Gamma L(2, 16).$$

We also have $|SL_2(16)| = 15 \cdot 16 \cdot 17$. Consider the cyclic 5-Sylow subgroup Q of $SL_2(16)$. If $SL_2(16)$ had a 5-complement, then $SL_2(16)$ would act faithfully on the 5 cosets of the complement, and hence $SL_2(16)$ would be a subgroup of S_5, which is impossible. Thus $SL_2(16)$ has no 5-complement. Now $[X : SL_2(16)]$ divides $[\Gamma L(2, 16) : SL_2(16)] = 4$, so $5 \nmid [X : SL_2(16)]$, therefore Q is the 5-Sylow subgroup of X. Suppose $Z \leq K \leq H \leq G$, where $K, H \triangle G$, $H/K \simeq SL_2(16)$ and $G/K \simeq X$. By Lemma 2, G/K has a 5-complement, say T/K. The second isomorphism theorem and $5 \nmid [G : H]$ imply that $H/K \cap T/K$ is a 5-complement for H/K, which is a contradiction.

Subcase 2. Assume G/Z has a nonabelian composition factor S having a 2-Sylow subgroup of order 8. There are five possibilities [5]:

(a) $S = PSL(2, q)$, q odd, $q > 3$.

(b) $S = A_7$.

(c) $S = J(11)$, the simple group discovered by Janko.

(d) $S = SL_2(8)$.

(e) S is a group of Ree type.

We rule out $PSL(2, q)$ and A_7 by Lemmas 1 and 3. Suppose $S = J(11)$. $\mathrm{Outer}(S) = 1$ [5] and $S \leq X \leq \mathrm{Aut}(S)$ by Lemma 1; hence $X = S$ where X is a homomorphic image of G/Z. But $|J(11)| = 2^3 \cdot 3 \cdot 5 \cdot 7 \cdot 11 \cdot 19$, which contradicts Lemma 2.

Next suppose $S = SL_2(8)$. $|S| = 7 \cdot 8 \cdot 9$. By Lemma 1, $S \leq X \leq \mathrm{Aut}(S)$, where $[\mathrm{Aut}(S) : S] = 3$. Thus $7 \nmid [X : S]$ and hence a cyclic 7-Sylow subgroup of S is also a 7-Sylow subgroup of X. X has a 7-complement by Lemma 2, and so S also has a 7-complement. S acts faithfully on the 7 cosets of the complement, so we have $S \leq S_7$. This is impossible as $SL_2(8)$ has a cyclic subgroup of order 9.

The remaining possibility is a group S of Ree type. Then $|S| = q^3(q-1)(q^3+1)$, where $q = 3^{2k+1}$, $k \geq 1$. S has a cyclic subgroup W of order $q - 1$ [13, p. 63]. By Lemma 5, there exists an odd prime p dividing

$|W|$ but not dividing $2k+1$. There is a cyclic p-Sylow subgroup P of W, and since $(q-1, q^3+1) = 2$, P is also a p-Sylow subgroup of S. By Lemma 1, $S \leq X \leq \mathrm{Aut}(S)$, where X is a homomorphic image of G/Z. $[\mathrm{Aut}(S):S] = 2k+1$ [10, 9.1]. Thus $p \nmid [X:S]$ and hence P is a cyclic p-Sylow subgroup of X. X has a p-complement by Lemma 2, and hence S has a p-complement. S acts faithfully on the $|P|$ cosets of the complement and $|P|$ divides $|W| = q-1$. Therefore S has a permutation representation of degree less than or equal to $q-1$, with the 1-character occurring only once. But in the character table of a group of Ree type [13, p. 87] there are no irreducible characters of degree less than or equal to $q-1$.

Subcase 3. Assume G/Z has a nonabelian composition factor S having 2-Sylow subgroup of order 4. This has been ruled out in Case 1.

Subcase 4. Assume G/Z has two nonabelian composition factors, S_1 and S_2, each having a 2-Sylow subgroup of order 4. Repeating the argument in Case 1 we see that $S_1 \simeq S_2 \simeq PSL(2, p^n)$ where $p^n > 3$ and p is an odd prime. $S_1 \leq X \leq \mathrm{Aut}(S_1)$ is not possible by Case 1, so by Lemma 1, $S_1 \times S_2 \leq X \leq \mathrm{Aut}(S_1 \times S_2)$, where X is a homomorphic image of G/Z. Let $\sigma \in \mathrm{Aut}(S_1 \times S_2)$. Using a Krull-Schmidt argument [11, 4.6.3], we can regard σ as a permutation on two letters, and the kernel of this action is $\mathrm{Aut}(S_1) \times \mathrm{Aut}(S_2)$. Therefore

$$\mathrm{Aut}(S_1 \times S_2)/\mathrm{Aut}(S_1) \times \mathrm{Aut}(S_2) \simeq C_2.$$

We consider two possibilities. First suppose $S_1 \times S_2 \leq X \leq \mathrm{Aut}(S_1) \times \mathrm{Aut}(S_2)$. By projecting onto $\mathrm{Aut}(S_1)$ we get

$$(S_1 \times S_2)\,\mathrm{Aut}(S_2)/\mathrm{Aut}(S_2) \leq X\,\mathrm{Aut}(S_2)/\mathrm{Aut}(S_2) \leq \mathrm{Aut}(S_1)$$

and so

$$(S_1 \times S_2)/(S_1 \times S_2) \cap \mathrm{Aut}(S_2) \leq X/\mathrm{Aut}(S_2) \cap X \leq \mathrm{Aut}(S_1).$$

But $(S_1 \times S_2) \cap \mathrm{Aut}(S_2) = S_2$, thus $S_1 \leq Y \leq \mathrm{Aut}(S_1)$ where Y is a homomorphic image of G/Z. Now apply Lemma 3 to get the desired contradiction. Since $[\mathrm{Aut}(S_1 \times S_2) : \mathrm{Aut}(S_1) \times \mathrm{Aut}(S_2)] = 2.$ the remaining possibility is when $X = \mathrm{Aut}(S_1 \times S_2)$. This is ruled out because the 2-Sylow subgroup of X would have order greater than 16. This completes the proof.

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