ON THE SOLVABILITY OF GROUPS OF CENTRAL TYPE

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ABSTRACT. Let $G$ be a finite group with center $Z$ and irreducible complex character $\chi$ so that $\chi(1)^2 = [G : Z]$. If the 2-Sylow subgroup of $G/Z$ has order 16 or less then $G$ is solvable.

Introduction. Let $G$ be a finite group with center $Z$. If $G$ has an irreducible complex character $\chi$ with $\chi(1) = [G : Z]$, then $G$ is called a group of central type $[4]$. It was conjectured in $[8]$ that groups of central type are solvable and several authors have given partial results in this direction $[2]$–$[4]$, $[6]$, $[9]$, and $[12]$. For example, in $[4]$ it is proved that if $G$ is a group of central type and if for any prime $p$, $p^m | [G : Z]$ implies $m \leq 2$ then $G$ is solvable. In $[6]$, the integer 2 in this result was replaced by 4. Here we show that if $2^m | [G : Z]$ implies $m \leq 4$ and $G$ is of central type, then $G$ is solvable.

The proof employs the well-known characterizations of simple groups with small 2-Sylow subgroups and information on possible homomorphic images of groups of central type given in $[6]$. All unexplained notation and terminology is as in $[7]$.

Lemma 1. Let $G$ be a finite group having a nonabelian composition factor $S$ appearing exactly $n$ times. Then there exist a homomorphic image $X$ of $G$ and an integer $m$ such that $1 \leq m \leq n$ and $S_1 \times \ldots \times S_m \leq X \leq Aut(S_1 \times \ldots \times S_m)$ where $S_i \cong S$ for all $i = 1, 2, \ldots, m$.

Proof. Use induction on $|G|$. Let $T$ be a minimal nontrivial normal subgroup of $G$. If $G$ is simple then $G = S$ and $S \leq G \leq Aut(S)$. So we may assume $|1| < T < G$. We have $T = T_1 \times \ldots \times T_k$ where the $T_i$ are isomorphic simple groups, $i = 1, 2, \ldots, k$.

Case 1. If $T_i \not\cong S$ for all $i = 1, 2, \ldots, k$, then consider $G_i = G/T$. By the inductive hypothesis there exists a homomorphic image $X$ of $G_1$ and an integer $m$ such that $1 \leq m \leq n$ and $S_1 \times \ldots \times S_m \leq X \leq Aut(S_1 \times \ldots \times S_m)$, where $S_i \cong S$. This is the desired homomorphic image $X$ of $G$.

Case 2. If $T_i \cong S$ for all $i = 1, 2, \ldots, k$, then $T \cong S_1 \times \ldots \times S_k$, $k \leq n$ and $S_i \cong S$. Since $T \triangleleft G$, $C_G(T) \triangleleft G$ and we have $G/C_G(T) \leq Aut(T)$. Since $T$ is minimal normal, $C_G(T) \cap T = \{1\}$ and thus

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Lemma 2. Let $G$ be a group of central type with a normal subgroup $K$ such that $G/K$ has a cyclic $p$-Sylow subgroup. Then $G/K$ has a $p$-complement. Furthermore, if $G/K$ is also simple then the prime $p$ is unique.

Proof. [6, 2.4(a) and its proof].

Lemma 3. Suppose $S \leq X \leq \text{Aut}(S)$, where $S = \text{PSL}(2, q)$, $q$ odd and $q > 3$, or $S = A_7$. Then $X$ is not the homomorphic image of a group of central type.

Proof. [6, 4.3 and the proof of 4.4].

The following number theoretic results are needed.

Lemma 4 (Zigmundy, 1896). Let $a$ and $n$ be integers both greater than one. Then there exists a prime divisor $p$ of $a^n - 1$ such that $p \nmid a^m - 1$ for all $m$ satisfying $1 < m < n$, except for the following cases:

(a) $n = 2$ and $a$ is a Meisenne number.
(b) $n = 6$ and $a = 2$.

Proof. [1].

Lemma 5. Let $a$ and $k$ be positive integers. Then there exists an odd prime $p$ such that $p \mid a^{2k+1} - 1$ and $p \nmid 2k + 1$.

Proof. By Lemma 4, there is a prime divisor $p$ of $a^{2k+1} - 1$ such that $p \nmid a^m - 1$ for all $1 < m < 2k + 1$. $p \mid a^{p-1} - 1$ by Fermat's little theorem. Hence $p - 1 \geq 2k + 1$, so $p > 2k + 1$. Therefore $p$ is odd and $p \nmid 2k + 1$.

Theorem 1. Let $G$ be a group of central type with center $Z$. If the 2-Sylow subgroup of $G/Z$ has order 16 or less, then $G$ is solvable.

Proof. If the 2-Sylow subgroup of $G/Z$ is trivial then $G/Z$ has odd order and hence is solvable. Since $[G:Z]$ is a square, the 2-Sylow subgroup of $G/Z$ has order 4 or 16. If $G/Z$ is nonsolvable, then it must have at least one nonabelian composition factor. There are several possibilities and these possibilities are considered by cases.

Case 1. Assume a 2-Sylow subgroup $P$ of $G/Z$ has order 4. Then $P = C_2 \times C_2$ by [4, Theorem 2 and Lemma 2]. If $G/Z$ has a nonabelian composition factor $S$ then its 2-Sylow subgroup must be $C_2 \times C_2$, since no nonabelian simple group can have $C_2$ as its 2-Sylow subgroup. Also $S$ appears exactly once since the 2-Sylow subgroup of $G/Z$ has order 4. Applying Lemma 1, $S \leq X \leq \text{Aut}(S)$ where $X$ is a homomorphic image of $G/Z$. Since $S$ is a simple group whose 2-Sylow subgroup is $C_2 \times C_2$, $S = \text{PSL}(2, p^n)$ where $p^n > 3$ and $p$ is an odd prime [5, 4.2.3]. So $\text{PSL}(2, p^n) \leq X \leq \text{Aut}(\text{PSL}(2, p^n))$.

By Lemma 3, $X$ is not a homomorphic image of a group of central type and this contradiction completes Case 1.
Case 2. Assume a 2-Sylow subgroup of $G/Z$ has order 16.

Subcase 1. Assume $G/Z$ has a nonabelian composition factor $S$ having a 2-Sylow subgroup $P$ of order 16. The only groups of order 16 which can occur as 2-Sylow subgroups of simple groups are the elementary abelian, the dihedral, and the semidihedral groups [5]. $P$ is of central type [4, Theorem 2] and so by [6, 3.2], the dihedral and semidihedral groups are eliminated since they have cyclic self-centralizing subgroups. Hence $P$ is elementary abelian of order 16. By [5, 4.2.3], $S = PSL(2, q), q \equiv \pm 3 \pmod{8}$ or $S = SL_2(16)$. The first possibility has already been ruled out in Case 1, so suppose $S = SL_2(16)$. By Lemma 1,

$$SL_2(16) \leq X \leq \text{Aut}(SL_2(16)) = \Gamma L(2, 16).$$

We also have $|SL_2(16)| = 15 \cdot 16 \cdot 17$. Consider the cyclic 5-Sylow subgroup $Q$ of $SL_2(16)$. If $SL_2(16)$ had a 5-complement, then $SL_2(16)$ would act faithfully on the 5 cosets of the complement, and hence $SL_2(16)$ would be a subgroup of $S_5$, which is impossible. Thus $SL_2(16)$ has no 5-complement. Now $[X : SL_2(16)]$ divides $[\Gamma L(2, 16) : SL_2(16)] = 4$, so $5 \nmid [X : SL_2(16)]$, therefore $Q$ is the 5-Sylow subgroup of $X$. Suppose $Z \leq K \leq H \leq G$, where $K, H \triangle G, H/K \simeq SL_2(16)$ and $G/K \simeq X$. By Lemma 2, $G/K$ has a 5-complement, say $T/K$. The second isomorphism theorem and $5 \nmid [G : H]$ imply that $H/K \cap T/K$ is a 5-complement for $H/K$, which is a contradiction.

Subcase 2. Assume $G/Z$ has a nonabelian composition factor $S$ having a 2-Sylow subgroup of order 8. There are five possibilities [5]:

(a) $S = PSL(2, q), q$ odd, $q > 3$.
(b) $S = A_7$.
(c) $S = J(11)$, the simple group discovered by Janko.
(d) $S = SL_2(8)$.
(e) $S$ is a group of Ree type.

We rule out $PSL(2, q)$ and $A_7$ by Lemmas 1 and 3. Suppose $S = J(11)$. $\text{Outer}(S) = 1$ [5] and $S \leq X \leq \text{Aut}(S)$ by Lemma 1; hence $X = S$ where $X$ is a homomorphic image of $G/Z$. But $|J(11)| = 2^3 \cdot 3 \cdot 5 \cdot 7 \cdot 11 \cdot 19$, which contradicts Lemma 2.

Next suppose $S = SL_2(8)$. $|S| = 7 \cdot 8 \cdot 9$. By Lemma 1, $S \leq X \leq \text{Aut}(S)$, where $[\text{Aut}(S) : S] = 3$. Thus $7 \nmid [X : S]$ and hence a cyclic 7-Sylow subgroup of $S$ is also a 7-Sylow subgroup of $X$. $X$ has a 7-complement by Lemma 2, and so $S$ also has a 7-complement. $S$ acts faithfully on the 7 cosets of the complement, so we have $S \leq S_7$. This is impossible as $SL_2(8)$ has a cyclic subgroup of order 9.

The remaining possibility is a group $S$ of Ree type. Then $|S| = q^3(q - 1)(q^2 + 1)$, where $q = 2^{2k+1}, k \geq 1$. $S$ has a cyclic subgroup $W$ of order $q - 1$ [13, p. 63]. By Lemma 3, there exists an odd prime $p$ dividing
|W| but not dividing \(2k + 1\). There is a cyclic \(p\)-Sylow subgroup \(P\) of \(W\), and since \((q - 1, q^3 + 1) = 2\), \(P\) is also a \(p\)-Sylow subgroup of \(S\). By Lemma 1, \(S \leq X \leq \text{Aut}(S)\), where \(X\) is a homomorphic image of \(G/Z\). \([\text{Aut}(S) : S] = 2k + 1\) [10, 9.1]. Thus \(p \nmid [X : S]\) and hence \(P\) is a cyclic \(p\)-Sylow subgroup of \(X\). \(X\) has a \(p\)-complement by Lemma 2, and hence \(S\) has a \(p\)-complement. \(S\) acts faithfully on the \(|P|\) cosets of the complement and \(|P|\) divides \(|W| = q - 1\). Therefore \(S\) has a permutation representation of degree less than or equal to \(q - 1\), with the 1-character occurring only once. But in the character table of a group of Ree type [13, p. 87] there are no irreducible characters of degree less than or equal to \(q - 1\).

Subcase 3. Assume \(G/Z\) has a nonabelian composition factor \(S\) having 2-Sylow subgroup of order 4. This has been ruled out in Case 1.

Subcase 4. Assume \(G/Z\) has two nonabelian composition factors; \(S_1\) and \(S_2\), each having a 2-Sylow subgroup of order 4. Repeating the argument in Case 1 we see that \(S_1 \cong S_2 \cong \text{PSL}(2, p^n)\) where \(p^n > 3\) and \(p\) is an odd prime. \(S_1 \leq X \leq \text{Aut}(S_1)\) is not possible by Case 1, so by Lemma 1, \(S_1 \times S_2 \leq X \leq \text{Aut}(S_1 \times S_2)\), where \(X\) is a homomorphic image of \(G/Z\). Let \(\sigma \in \text{Aut}(S_1 \times S_2)\). Using a Krull-Schmidt argument [11, 4.6.3], we can regard \(\sigma\) as a permutation on two letters, and the kernel of this action is \(\text{Aut}(S_1) \times \text{Aut}(S_2)\). Therefore

\[
\text{Aut}(S_1 \times S_2)/\text{Aut}(S_1) \times \text{Aut}(S_2) \cong C_2.
\]

We consider two possibilities. First suppose \(S_1 \times S_2 \leq X \leq \text{Aut}(S_1) \times \text{Aut}(S_2)\).

By projecting onto \(\text{Aut}(S_1)\) we get

\[
(S_1 \times S_2) \text{Aut}(S_2)/\text{Aut}(S_2) \leq X \text{Aut}(S_2)/\text{Aut}(S_2) \leq \text{Aut}(S_1).
\]

and so

\[
(S_1 \times S_2)/(S_1 \times S_2) \cap \text{Aut}(S_2) \leq X/\text{Aut}(S_2) \cap X \leq \text{Aut}(S_1).
\]

But \((S_1 \times S_2) \cap \text{Aut}(S_2) = S_2\), thus \(S_1 \leq Y \leq \text{Aut}(S_1)\) where \(Y\) is a homomorphic image of \(G/Z\). Now apply Lemma 3 to get the desired contradiction. Since \([\text{Aut}(S_1 \times S_2) : \text{Aut}(S_1) \times \text{Aut}(S_2)] = 2\), the remaining possibility is when \(X = \text{Aut}(S_1 \times S_2)\). This is ruled out because the 2-Sylow subgroup of \(X\) would have order greater than 16. This completes the proof.

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BIBLIOGRAPHY


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