PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 52, October 1975

CODIMENSION OF COMPACT M-SEMILATTICES

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ABSTRACT. This paper is a generalization of [5] and gives a partial answer to Question 31 in [1], i.e., if S is a compact M-semilattice of finite codimension and $x \neq y$, then there exists a closed subsemilattice A of S such that A separates x and y in S and $\mathrm{cd}\, A < \mathrm{cd}\, S$.

A topological semilattice is a Hausdorff space with an associative continuous operation which is commutative and idempotent. We denote

$$M(x) = \{y \in S \mid x \le y\}, \quad L(x) = \{y \in S \mid y \le x\}, \quad [x, y] = M(x) \cap L(y),$$

and $F(A)$ the boundary of A in S. The breadth of S (Br S) is the smallest integer n such that any finite subset F of S contains a subset G of at most n elements such that $\inf F = \inf G$. The codimension of S is the smallest integer n such that $i^*\colon H^n(S) \to H^n(A)$ is onto for each closed subset A and inclusion i.

I. **Lemma 1.1.** *If S is a topological semilattice and $a \in S$, then $F(M(a))$ is an ideal of $M(a)$ if $F(M(a)) \neq \varnothing$.*

Proof. Let $p \in F(M(a))$ and $q \in M(a)$. Suppose pq belongs to the interior of $M(a)$. Then there exists an open set U of p such that $Uq \subset M(a)$. For each $z \in U$, $zq \in M(a)$ implies $z \in M(a)$. Hence $p \in U$ which is contained in $M(a)$, contrary to $p \in F(M(a))$.

A topological semilattice S is said to be an *M-semilattice* if $M(x)$ is connected for each $x \in S$.

Lemma 1.2. *If S is an M-semilattice, then $F(M(a))$ is an M-semilattice if nonempty.*

Proof. Let $x, y \in F(M(a))$ and $x < y$. Then $M(x)$ is connected. Since $F(M(a))$ is an ideal of $M(a)$, then $yM(x) \subset F(M(a)) \cap L(y) \cap M(x)$ and also $yM(x)$ contains x and y.

Lemma 1.3. *If S is a compact M-semilattice, then S is locally connected.*

Proof. Let $x \in W$ and W be open in S. We can assume W convex since S is a compact partially ordered space. Let $x \in V$ and V open in S and $V^2 \subset W$. For all $p, q \in V$, $pq \in V^2$. Then there exist arc-chains A and B from pq to p and pq to q respectively. Since W is convex, then A and B

Received by the editors February 2, 1974 and, in revised form, August 13, 1974.
AMS (MOS) subject classifications (1970). Primary 22A15; Secondary 06A20, 54F05.

are contained in W. Hence S is connected im kleinem at each $x \in S$; S is locally connected.

Theorem 1. *If S is a compact M-semilattice, then $F(M(a))$ is a compact connected locally connected semilattice and if S has finite codimension, then S has a basis of closed neighborhoods each of which is a subsemilattice.*

Proof. Apply Lemma 1.3 and Theorem 3.4 of [2].

II. If S is a compact semilattice with identity, we can define

$$a \vee b = \inf\{x \mid a \le x \text{ and } b \le x\}$$

which exists and is the least upper bound for a and b.

Lemma 2.1. *Let S be a compact semilattice and $\{x_\alpha \mid \alpha \in D\}$ is an increasing net in S. Then x_α converges to x if and only if $x = \operatorname{lub}\{x_\alpha \mid \alpha \in D\}$.*

Proof. Suppose x_α converges to x. Let $\beta \in D$. Then $x_\beta x_\alpha$ converges to $x_\beta x$. But $x_\beta \le x_\alpha$ residually. Hence $x_\beta x = x_\beta$. Thus $x_\beta \le x$, i.e., $\operatorname{lub}\{x_\alpha \mid \alpha \in D\} \le x$. Suppose $x \not\le \operatorname{lub} x_\alpha$. Then there are open sets $x \in U$, $\operatorname{lub} x_\alpha \in V$ such that $(U \times V) \cap \le \, = \emptyset$. Choose any $x_\beta \in U$. Then $x_\beta \le \operatorname{lub} x_\alpha$, a contradiction. Hence $x = \operatorname{lub} x_\alpha$.

Suppose $x = \operatorname{lub} x_\alpha$. Let $x \in U$ open in S. Suppose there exists a cofinal subset E of D such that $\{x_\alpha \mid \alpha \in E\} \cap U = \emptyset$. Then there exists a subnet y_β of $\{x_\alpha \mid \alpha \in E\}$ such that y_β converges to some $y \notin U$. By the previous paragraph, $y = \operatorname{lub} y_\beta = \operatorname{lub} x_\alpha = x$. We have a contradiction.

Lemma 2.2. *If S is a compact semilattice and x_α is an increasing net converging to x, then $a \vee x_\alpha$ is an increasing net converging to $a \vee x$.*

Proof. Since $a \vee x_\alpha \le a \vee x$ for each α, then $\operatorname{lub}(a \vee x_\alpha) \le a \vee x$. Also for each α, $x_\alpha \le a \vee x_\alpha$. Hence $x \le \operatorname{lub}(a \vee x_\alpha)$ which yields $a \vee x \le \operatorname{lub}(a \vee x_\alpha)$.

Lemma 2.3. *If S is a compact M-semilattice with identity, then* $\operatorname{cd} S = \operatorname{Br} S$.

Proof. A generalization of Corollary 2.4 of [4].

Theorem 2. *If S is a compact M-semilattice of positive codimension n, then* $\operatorname{cd}(F(M(a))) < n$.

Proof. We first show that if $a \ne x \in F(M(a))$, then $\operatorname{cd}[a, x] < n$. Choose a closed neighborhood V of x such that $V \cap L(a) = \emptyset$ and $V^2 = V$. Since $x \in F(M(a))$, then $U \cap S \backslash M(a) \ne \emptyset$ for each open set U containing x. Hence there is a net $\{x_\alpha\} \subset V$ such that $x_\alpha \notin M(a)$ and x_α converges to x. Let $y_\alpha = \inf\{x_\beta \mid \alpha \le \beta\}$. Then $y_\alpha \in V \backslash M(a)$. If W is a closed neighborhood of x

and $W^2 = W$, then there exists α such that for each $\beta \geq \alpha$, $x_\beta \in W$. Then $\inf\{x_\beta \mid \beta \geq \gamma\} \in W$ for all $\gamma \geq \alpha$. Hence y_α is an increasing net converging to x.

For each $\alpha \in D$, $[a, a \vee y_\alpha]$ has breadth less than n [6, Lemma 1.1] since $\operatorname{Br} L(x) = \operatorname{cd} L(x) \leq n$. Since $\{[a, a \vee y_\alpha]\}$ is a chain of subsemilattices, then

$$\operatorname{Br}(\bigcup [a, a \vee y_\alpha]) = \operatorname{Br}((\bigcup [a, a \vee y_\alpha])^*) < n,$$

where * denotes closure in S. Since y_α converges to x, then $a \vee y_\alpha$ converges to $a \vee x$ and $(a \vee y_\alpha)M(a)$ converges to $(a \vee x)M(a)$ in terms of lim inf and lim sup. Since $[a, a \vee y_\alpha] = (a \vee y_\alpha)M(a)$ and $[a, x] = [a, a \vee x] = (a \vee x)M(a)$ and $[a, a \vee y_\alpha] \subset [a, a \vee x]$, then $[a, x] = (\bigcup [a, a \vee y_\alpha])^*$. Thus $\operatorname{Br}[a, x] < n$. Since $[a, x]$ is a compact M-semilattice with identity, then $\operatorname{cd}[a, x] = \operatorname{Br}[a, x] < n$.

Suppose $F(M(a)) \neq \emptyset$. Then by Theorem 1 and [3, Corollary 2], $F(M(a))$ has a point x such that $\operatorname{cd}(F(M(a))) = \operatorname{cd}(F(M(a)) \cap L(x))$. But

$$\operatorname{cd}(F(M(a)) \cap L(x)) \leq \operatorname{cd}[a, x] < n.$$

Theorem 3. *If S is a compact M-semilattice and $x \neq y$, then there exists a closed subsemilattice A of S such that A separates x and y in S and* $\operatorname{cd} A < \operatorname{cd} S$.

Proof. Suppose $x \neq y$. Then $x \not\leq y$ or $y \not\leq x$. Assume $x \not\leq y$. There exist open sets U, V containing x and y such that $(U \times V) \cap \leq\, = \emptyset$. Let K be a closed neighborhood of x contained in U and $K^2 = K$. Choose $a = \inf K$. Then x belongs to the interior of $M(a)$ and $y \notin M(a)$. Hence $F(M(a))$ separates x and y. Also $\operatorname{cd} F(M(a)) < \operatorname{cd} S$ by Theorem 2.

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