CODIMENSION OF COMPACT M-SEMILATTICES

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ABSTRACT. This paper is a generalization of [5] and gives a partial answer to Question 31 in [1], i.e., if \( S \) is a compact \( M \)-semilattice of finite codimension and \( x \neq y \), then there exists a closed subsemilattice \( A \) of \( S \) such that \( A \) separates \( x \) and \( y \) in \( S \) and \( \text{cd} A < \text{cd} S \).

A topological semilattice is a Hausdorff space with an associative continuous operation which is commutative and idempotent. We denote

\[ M(x) = \{ y \in S \mid x \leq y \}, \quad L(x) = \{ y \in S \mid y \leq x \}, \quad [x, y] = M(x) \cap L(y), \]

and \( F(A) \) the boundary of \( A \) in \( S \). The breadth of \( S \) (\( \text{Br} S \)) is the smallest integer \( n \) such that any finite subset \( F \) of \( S \) contains a subset \( G \) of at most \( n \) elements such that \( \inf F = \inf G \). The codimension of \( S \) is the smallest integer \( n \) such that \( i^* : H^n(S) \to H^n(A) \) is onto for each closed subset \( A \) and inclusion \( i \).

I. Lemma 1.1. If \( S \) is a topological semilattice and \( a \in S \), then \( F(M(a)) \) is an ideal of \( M(a) \) if \( F(M(a)) \neq \emptyset \).

Proof. Let \( p \in F(M(a)) \) and \( q \in M(a) \). Suppose \( pq \) belongs to the interior of \( M(a) \). Then there exists an open set \( U \) of \( p \) such that \( Uq \subset M(a) \). For each \( z \in U \), \( zq \in M(a) \) implies \( z \in M(a) \). Hence \( p \in U \) which is contained in \( M(a) \), contrary to \( p \in F(M(a)) \).

A topological semilattice \( S \) is said to be an \( M \)-semilattice if \( M(x) \) is connected for each \( x \in S \).

Lemma 1.2. If \( S \) is an \( M \)-semilattice, then \( F(M(a)) \) is an \( M \)-semilattice if nonempty.

Proof. Let \( x, y \in F(M(a)) \) and \( x < y \). Then \( M(x) \) is connected. Since \( F(M(a)) \) is an ideal of \( M(a) \), then \( yM(x) \subset F(M(a)) \cap L(y) \cap M(x) \) and also \( yM(x) \) contains \( x \) and \( y \).

Lemma 1.3. If \( S \) is a compact \( M \)-semilattice, then \( S \) is locally connected.

Proof. Let \( x \in W \) and \( W \) be open in \( S \). We can assume \( W \) convex since \( S \) is a compact partially ordered space. Let \( x \in V \) and \( V \) open in \( S \) and \( V^2 \subset W \). For all \( p, q \in V \), \( pq \in V^2 \). Then there exist arc-chains \( A \) and \( B \) from \( pq \) to \( p \) and \( pq \) to \( q \) respectively. Since \( W \) is convex, then \( A \) and \( B \)
are contained in \( W \). Hence \( S \) is connected in kleinem at each \( x \in S \); \( S \) is locally connected.

**Theorem 1.** If \( S \) is a compact \( M \)-semilattice, then \( F(M(a)) \) is a compact connected locally connected semilattice and if \( S \) has finite codimension, then \( S \) has a basis of closed neighborhoods each of which is a subsemilattice.

**Proof.** Apply Lemma 1.3 and Theorem 3.4 of \([2]\).
and $W^2 = W$, then there exists $a$ such that for each $\beta \geq a$, $x_\beta \in W$. Then
\[
\inf \{x_\beta \mid \beta \geq y\} \in W \text{ for all } y \geq a.
\]
Hence $y_a$ is an increasing net converging to $x$.

For each $a \in D$, $[a, a \lor y_a]$ has breadth less than $n$ [6, Lemma 1.1] since $\text{Br} L(x) = \text{cd} L(x) \leq n$. Since $\{[a, a \lor y_a]\}$ is a chain of subsemilattices, then
\[
\text{Br}(\bigcup [a, a \lor y_a]) = \text{Br}(\bigcup [a, a \lor y_a])^* < n,
\]
where $^*$ denotes closure in $S$. Since $y_a$ converges to $x$, then $a \lor y_a$ converges to $a \lor x$ and $(a \lor y_a)\text{M}(a)$ converges to $(a \lor x)\text{M}(a)$ in terms of lim inf and lim sup. Since $[a, a \lor y_a] = (a \lor y_a)\text{M}(a)$ and $[a, x] = [a, a \lor x] = (a \lor x)\text{M}(a)$ and $[a, a \lor y_a] \subset [a, a \lor x]$, then $[a, x] = (\bigcup [a, a \lor y_a])^*$.

Thus $\text{Br}[a, x] < n$. Since $[a, x]$ is a compact $M$-semilattice with identity, then $\text{cd}[a, x] = \text{Br}[a, x] < n$.

Suppose $F(M(a)) \neq \emptyset$. Then by Theorem 1 and [3, Corollary 2], $F(M(a))$ has a point $x$ such that $\text{cd}(F(M(a))) = \text{cd}(F(M(a)) \cap L(x))$. But
\[
\text{cd}(F(M(a)) \cap L(x)) \leq \text{cd}[a, x] < n.
\]

**Theorem 3.** If $S$ is a compact $M$-semilattice and $x \neq y$, then there exists a closed subsemilattice $A$ of $S$ such that $A$ separates $x$ and $y$ in $S$ and $\text{cd} A < \text{cd} S$.

**Proof.** Suppose $x \neq y$. Then $x \subsetneq y$ or $y \subsetneq x$. Assume $x \subsetneq y$. There exist open sets $U, V$ containing $x$ and $y$ such that $(U \times V) \cap \leq = \emptyset$. Let $K$ be a closed neighborhood of $x$ contained in $U$ and $K^2 = K$. Choose $a = \inf K$. Then $x$ belongs to the interior of $M(a)$ and $y \notin M(a)$. Hence $F(M(a))$ separates $x$ and $y$. Also $\text{cd} F(M(a)) < \text{cd} S$ by Theorem 2.

**REFERENCES**


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