

## ANNULAR FUNCTIONS IN PROBABILITY

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**ABSTRACT.** A function  $f$  holomorphic in the open unit disk  $U$  is said to be strongly annular if there exists a sequence  $\{C_n\}$  of concentric circles converging outward to the boundary of  $U$  such that the minimum of  $|f|$  on  $C_n$  tends to infinity as  $n$  increases. We show here that such functions with Maclaurin coefficients  $\pm 1$  form a residual set in the space of functions with coefficients  $\pm 1$ . We also show that the set of  $t$  in  $[0, 1]$  for which  $\sum r_n(t)z^n$  is strongly annular ( $r_n$  is the  $n$ th Rademacher function) is residual, and measurable with measure either 0 or 1.

Let  $U$  denote the open unit disk, and  $H(U)$  the space of holomorphic functions in  $U$ , where the topology on  $H(U)$  is the (complete, metric) topology of almost uniform convergence [4, pp. 142–148]. A function  $f$  in  $H(U)$  is said to be *annular* if there exists a sequence  $\{J_n\}$  of closed Jordan curves about 0, converging outward to the boundary  $T$  of  $U$  such that the minimum of  $|f|$  on  $J_n$  approaches infinity as  $n \rightarrow \infty$ . If the  $J_n$  can be taken as circles concentric with  $T$ ,  $f$  is *strongly annular*. In [5], Kahane shows (using different terminology) that if  $\{Z_n\}$  is a complex normal sequence of random variables, and if  $a_n = n^\alpha$  ( $\alpha > 1/2$ ), then the function  $F_t(z) = \sum a_n Z_n(t)z^n$  is a.s. strongly annular. As a consequence, the image of  $F$  fills a.s. the whole plane. It is not known if the latter holds for  $a_n = n^\alpha$ ,  $-1/2 < \alpha \leq 1/2$ . Since in [2], Bonar and Carroll give a ‘hard analysis’ construction of a strongly annular function whose Maclaurin coefficients are all either  $\pm 1, 0$ , it is natural to let  $\alpha = 0$  in the above, and to replace  $\{Z_n\}$  with the sequence  $\{r_n\}$  of Rademacher functions. In this paper we show

**Theorem 2.** *The set  $S^* = \{t \in [0, 1]: G_t(z) \text{ is strongly annular}\}$  is residual in  $[0, 1]$ , where  $G_t(z) = \sum_{\nu=0}^{\infty} r_\nu(t)z^\nu$ , and*

**Theorem 1.** *The set  $S = \{f \in E: f \text{ is strongly annular}\}$  is residual in  $E$ , where  $E = \{f \in H(U): f(z) = \sum a_n z^n, a_n = \pm 1\}$ .*

We remark that although  $E$  is not a topological vector space, it is a

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complete metric space with the relative topology inherited from  $H(U)$ . Although Theorem 2 follows from the proof of Theorem 1, we list it separately because its proof is nontrivial, and because it connects with the probabilistic considerations mentioned in the remarks of this paper.

To prove Theorem 1, we use a method of Bonar and Carroll [1]. First, we need

**Lemma 1.1.** *For every positive real number  $R$ , there exists a polynomial*

$$P(R; z) = \sum_{\nu=0}^{N(R)} \alpha(R; \nu) z^\nu$$

where  $\alpha(R; \nu) = \pm 1$ ,  $\nu = 0, 1, \dots, N(R)$ , and  $\min_{z \in T} |P(R; z)| > R$ .

**Proof.** The proof is given by Carroll, Eustice, and Figiel [3]. For completeness, we reproduce their ideas here:

If  $P$  is a polynomial of degree  $k$  with coefficients  $\pm 1$ , then  $Q(z) = P(z)P(z^{k+1})$  also has such coefficients. Thus, it suffices to produce a polynomial  $P$  with coefficients  $\pm 1$  whose minimum modulus on  $T$  is greater than 1. Now  $P(z) = z^4 + z^3 + z^2 - z + 1$  is such a polynomial, since  $|P(e^{i\theta})|^2 = 5 + 2 \cos 4\theta + 2 \cos 2\theta$ .

Let us proceed to the proof of Theorem 1. We recall that a basic open neighborhood of a function  $f$  in  $H(U)$  is given by

$$V(f, K, \epsilon) = \left\{ g \in H(U) : \max_{z \in K} |f(z) - g(z)| < \epsilon \right\}$$

where  $K$  is a compact subset of  $U$ . For  $n = 2, 3, \dots$  we let

$$S_n = \{f \in E : \text{there exists } r = r(f) \text{ with } 1 - 1/n < r < 1, \\ \text{such that } |f(z)| > n \text{ for all } |z| = r\}.$$

It is clear that  $S = \bigcap_{n=2}^{\infty} S_n$ ; we shall show that each  $S_n$  is open and dense in  $E$ .

Suppose that  $f$  is in  $S_n$ , so that there exists a real number  $r$ ,  $1 - 1/n < r < 1$  such that  $|f(z)| > n$  for all  $|z| = r$ . Taking  $K$  to be that circle, and  $\epsilon = \min_{z \in K} |f(z)| - n$ , we see that all of  $V(f, K, \epsilon) \cap E$  lies in  $S_n$ .

Proceeding to the density, we let

$$(1.1) \quad f(z) = \sum_{\nu=0}^{\infty} a_\nu z^\nu, \quad a_\nu = \pm 1,$$

be an arbitrary function in  $E$ , and we let  $V(f, K, \epsilon) \cap E$  be an arbitrary basic open neighborhood of  $f$ . We may choose  $\lambda$  large so that for all  $z \in K$ ,

$$(1.2) \quad \sum_{\nu=\lambda+1}^{\infty} |z|^\nu < \epsilon/2.$$

We let

$$B = \max_{z \in T} \left| \sum_{\nu=0}^{\lambda} a_{\nu} z^{\nu} \right|,$$

and, using the notation of Lemma 1.1, we set

$$P(2^{\lambda+3}(B+n); z) = P(z)$$

where  $\deg P(z) = s$ . We may select  $r$ ,  $1 - 1/n < r < 1$ , such that for  $|z| = r$ ,

$$(1.3) \quad |P(z)| > 2^{\lambda+2}(B+n).$$

Finally we define  $g(z) \in E$  by

$$(1.4) \quad g(z) = \sum_{\nu=0}^{\lambda} a_{\nu} z^{\nu} + \left( \frac{P(z)}{1 - z^{s+1}} \right) z^{\lambda+1}.$$

Combining this with (1.1) and (1.2), we see that  $g$  lies in  $V(f, K, \epsilon) \cap E$ .

Since  $1 - 1/n < r < 1$ , we see by (1.3) and (1.4) that for  $|z| = r$ ,

$$|g(z)| > \frac{2^{\lambda+2}(B+n)}{2} \left( \frac{1}{2} \right)^{\lambda+1} - B = n,$$

so that  $g$  is also in  $S_n^*$ .  $\square$

In what follows, we let  $D$  be the dyadic rationals and

$$S_n^* = \{t \in [0, 1]: \text{there exists } r = r(t) \text{ with } 1 - 1/n < r < 1, \\ \text{such that } |G_t(z)| > n \text{ for all } |z| = r\},$$

where  $G$  is defined as in Theorem 2.

**Lemma 2.1.** *If  $x \in ([0, 1] - D) \cap S_n^*$ , there exists  $\epsilon > 0$  ( $\epsilon = \epsilon(x)$ ) such that  $(x - \epsilon, x + \epsilon) \subseteq S_n^*$ .*

**Proof.** Let  $x \in ([0, 1] - D) \cap S_n^*$ , so that for some  $r$ ,  $1 - 1/n < r < 1$ , we have that  $|G_x(z)| > n$  for all  $|z| = r$ . Thus, there exists a positive number  $k$  and an integer  $N$  such that  $|\sum_{\nu=0}^R r_{\nu}(x)z^{\nu}| > n + k$  for all  $R > N$ ,  $|z| = r$ . We may choose  $\lambda$  large so that  $\lambda > N$ , and  $\sum_{\nu=\lambda+1}^{\infty} |z|^{\nu} < k$  for all  $|z| = r$ . Since  $x \notin D$ , we may choose  $\epsilon$  small such that if  $|x - t| < \epsilon$ , then  $r_{\nu}(x) = r_{\nu}(t)$  for  $\nu = 0, 1, \dots, \lambda$ . For all such  $t$ , we have for  $|z| = r$  that

$$|G_t(z)| = \left| \sum_{\nu=0}^{\lambda} r_{\nu}(x)z^{\nu} + \sum_{\nu=\lambda+1}^{\infty} r_{\nu}(t)z^{\nu} \right| > n + k - k = n.$$

**Lemma 2.2.** *If  $x \in [0, 1]$ , and  $\delta > 0$  is given, there exists  $y \in ([0, 1] - D) \cap S_n^*$  such that  $|x - y| < \delta$ .*

**Proof.** We remark that if  $y \in [0, 1] - D$ ,  $y$  may be written uniquely as

$$y = \sum_{\nu=0}^{\infty} \beta_{\nu} 2^{-(\nu+1)}, \quad \sum_{\nu=0}^{\infty} \beta_{\nu} = \infty, \quad \text{where } \beta_{\nu} = \beta_{\nu}(y) = 0 \text{ or } 1.$$

If  $y$  is written this way, then we have that  $r_{\nu}(y) = 1 - 2\beta_{\nu}(y)$ .

We recall now that in the proof of Theorem 1, we were able to construct a function  $g(z) \in S_n$  whose first  $\lambda$  Maclaurin coefficients agreed with an arbitrary function  $f(z) \in E$ .  $g(z)$  was of the form (1.4), where  $P(z)$  was a polynomial of large minimum modulus on  $T$ . We remark also that since  $P(z)$  had coefficients equal to 1 and  $-1$ , the function  $g(z)$  had infinitely many coefficients equal to  $\pm 1$ .

Let  $x \in [0, 1]$ , and  $\delta > 0$  be given. We choose  $\lambda$  large such that  $\sum_{\nu=\lambda+1}^{\infty} 2^{-(\nu+1)} < \delta$ , and consider the function  $f(z) = \sum_{\nu=0}^{\infty} r_{\nu}(x) z^{\nu} \in E$ . As in Theorem 1, we may construct a function  $g(z) \in S_n$  such that  $g(z) = \sum_{\nu=0}^{\infty} a_{\nu} z^{\nu}$ , where, for  $\nu = 0, 1, \dots, \lambda$ ,  $a_{\nu} = r_{\nu}(x)$ , and infinitely many of the  $a_{\nu}$  are equal to  $\pm 1$ . We then define  $y \in [0, 1] - D$  by

$$y = \sum_{\nu=0}^{\infty} \left( \frac{1 - a_{\nu}}{2} \right) 2^{-(\nu+1)}.$$

Since

$$x = \sum_{\nu=0}^{\infty} \left( \frac{1 - r_{\nu}(x)}{2} \right) 2^{-(\nu+1)}$$

we have that

$$|x - y| \leq \sum_{\nu=\lambda+1}^{\infty} 2^{-(\nu+1)} < \delta.$$

Finally, we see that  $y \in S_n^*$  since  $\sum r_{\nu}(y) z^{\nu} = g(z)$ .

To prove Theorem 2, we note that  $S^* = \bigcap_{n=2}^{\infty} S_n^*$ , and that by Lemmas 2.1 and 2.2 each  $S_n^*$  contains a dense open subset of  $[0, 1]$ —namely  $U\{(x - \epsilon, x + \epsilon) : x \in ([0, 1] - D) \cap S_n^*\}$ .

**Remarks.**  $S^*$  is measurable, since by Lemma 2.1,

$$S_n^* = U\{(x - \epsilon, x + \epsilon) : x \in ([0, 1] - D) \cap S_n^*\} \cup \{x : x \in D \cap S_n^*\}.$$

Since  $S^*$  is a tail event, we have by the zero-one law [5, p. 6], that its measure is either 0 or 1. Because of Theorem 2, it would in some sense be "expected" that the measure of  $S^*$  is 1. It is not known, however, if this is the case, and since  $S_n^*$  is not a tail event, one may not use the zero-one law to reduce the problem of proving that the measure of  $S^*$  is 1 to that of merely showing that  $S_n^*$  has positive measure.

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