PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 52, October 1975

SUBORDINATION BY CONVEX FUNCTIONS

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ABSTRACT. The following theorem is proven: *Let* $F(z)$ *be convex and univalent in* $\Delta = \{z: |z| < 1\}$, $F(0) = 1$. *Let* $f(z)$ *be analytic in* Δ, $f(0) = 1$, $f'(0) = \dots = f^{(n-1)}(0) = 0$, *and let* $f(z) \prec F(z)$ *in* Δ. *Then for all* $\gamma \neq 0$, $\operatorname{Re}\gamma \geq 0$,

$$\gamma z^{-\gamma} \int_0^z \tau^{\gamma-1} f(\tau)\, d\tau \prec \gamma z^{-\gamma/n} \int_0^{z^{1/n}} \tau^{\gamma-1} F(\tau^n)\, d\tau.$$

This theorem, in combination with a method of D. Styer and D. Wright, leads to the following

Corollary. *Let* $f(z)$, $g(z)$ *be convex univalent in* Δ, $f(0) = f''(0) = g(0) = g''(0) = 0$. *Then* $f(z) + g(z)$ *is starlike univalent in* Δ.

Other applications of the theorem are concerned with the subordination of $f(z)/z$ where $f(z)$ belongs to certain classes of convex univalent functions.

Introduction. In this paper we are concerned with subordination results for special classes of convex univalent functions. Let $\Delta_r = \{z: |z| < r\}$ and $\Delta_1 = \Delta$. We recall the definition of subordination between two functions, say f and F, analytic in Δ. This means that $f(0) = F(0)$ and there is an analytic function $\phi(z)$ so that $\phi(0) = 0$, $|\phi(z)| < 1$ and $f(z) = F(\phi(z))$. This relation shall be denoted by $f \prec F$. If F is univalent in Δ the subordination is equivalent to $f(0) = F(0)$ and range $f(z) \subset$ range $F(z)$.

One of our main tools is the following result of I. S. Jack, which, in fact, is a modification of Julia's theorem [2, p. 28].

Lemma 1. *Let* $w(z)$ *be analytic in* Δ_R, $w^{(k)}(0) = 0$, $0 \leq k \leq n$. *Then if* $|w(z)|$ *attains its maximum value on the circle* $|z| = r < R$ *at* z_0, *we have*

$$\rho = z_0 w'(z_0)/w(z_0) \geq n + 1.$$

Another main tool is the convolution theorem of T. Sheil-Small and the second author. We recall that the Hadamard convolution of two functions $f(z) = \sum_{n=1}^\infty a_n z^n$ and $g(z) = \sum_{n=1}^\infty b_n z^n$ is denoted by $f * g(z)$ and defined by $f * g(z) = \sum_{n=1}^\infty a_n b_n z^n$. In [6] the authors prove that the convolution of two convex univalent functions is convex and univalent. A function $f(z)$ analytic in Δ with $f(0) = 0$, $f'(0) = 1$, is said to be convex of order α, $0 \leq \alpha < 1$,

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Received by the editors August 19, 1974.
AMS (MOS) subject classifications (1970). Primary 30A32, 30A40.
Key words and phrases. Subordination, convolution, convex.
[1] The research of the first author was, in part, supported by a University of Delaware Faculty Research Summer Grant.

if $\operatorname{Re}(1 + zf''(z)/f'(z)) > \alpha$ for z in Δ. We denote this class by $K(\alpha)$. For any set of complex numbers A, we let $\overline{A}$ denote the closure and ∂A denote the boundary.

Subordination theorems.

Lemma 2. *Let $F(z)$ be convex univalent in Δ, $F(0) = 1$. Let $f(z)$ be analytic in Δ, $f(0) = 1$, and suppose $f(z) \prec F(z)$ in Δ. Then for all $\gamma \neq 0$ with $\operatorname{Re}\gamma \geq 0$, we have $g(z) \prec F(z)$ where*

$$g(z) = \gamma z^{-\gamma} \int_0^z \tau^{\gamma-1} f(\tau)\, d\tau.$$

Proof. We have $g(z) = f * h(z)$ where

$$h(z) = \gamma z^{-\gamma} \int_0^z \frac{\tau^{\gamma-1}}{1-\tau}\, d\tau. \tag{1}$$

We prove, by Lemma 1, that $\operatorname{Re} h(z) > \frac{1}{2}$ in Δ.

Let $h(z) = 1/(1 - w(z))$, where $w(0) = 0$. Clearly $w(z)$ is meromorphic in all of Δ and analytic in Δ_R where $R = \min\{|z_k|: h(z_k) = 0\}$. Let z^* denote a point on $|z| = R$ with $h(z^*) = 0$. z^* is a pole of $w(z)$ and so there is a neighborhood of z^* in which $|w(z)| > 1$ holds. This implies that there is a point z_0, $|z_0| < R$, with $|w(z_0)| = 1$ and $|w(z)| \leq 1$ for $|z| \leq |z_0|$. Then differentiating (1), we find

$$-\gamma h(z)/z + \gamma/z(1 - z) = w'(z)/(1 - w(z))^2,$$

or equivalently

$$\frac{1}{1-z} = \frac{1}{\gamma}\frac{zw'(z)}{w(z)}\frac{w(z)}{(1-w(z))^2} + \frac{1}{1-w(z)}.$$

This implies for $z = z_0$ that $\operatorname{Re}(1 - z_0)^{-1} \leq \frac{1}{2}$, since $z_0 w'(z_0)/w(z_0) = \rho \geq 1$ by Lemma 1 and $w(z_0)/(1 - w(z_0))^2$ is a real number less than $-\frac{1}{4}$. Hence this implies that $R \geq 1$ and $\operatorname{Re} h(z) > \frac{1}{2}$ in Δ. Consequently, by the Herglotz formula, there exists a probability measure $\mu(\theta)$ such that

$$h(z) = \int_0^{2\pi} \frac{du(\theta)}{1 - ze^{-i\theta}} \quad \text{and} \quad g(z) = \int_0^{2\pi} f(ze^{-i\theta})\, d\mu(\theta),$$

which implies the result.

Theorem 1. *Let $F(z)$ be convex univalent in Δ, $F(0) = 1$. Let $f(z)$ be analytic in Δ, $f(0) = 1$, $f'(0) = \cdots = f^{(n-1)}(0) = 0$, and let $f(z) \prec F(z)$ in Δ. Then for all $\gamma \neq 0$, $\operatorname{Re}\gamma \geq 0$,*

$$g(z) \equiv \gamma z^{-\gamma} \int_0^z \tau^{\gamma-1} f(\tau)\, d\tau \prec \gamma z^{-\gamma/n} \int_0^{z^{1/n}} \tau^{\gamma-1} F(\tau^n)\, d\tau \equiv G(z). \tag{2}$$

Proof. We have $G(z) = F(z) * \sum_{j=0}^{\infty} \gamma z^j/(nj + \gamma)$. It follows from a result of the second author [5] that the second factor is convex univalent in Δ. In fact, any function $\sum_{j=0}^{\infty} z^j/(j + \gamma)$ where $\operatorname{Re}\gamma \geq 0$ has this property. The convolution theorem [6] implies that $G(z)$ is convex univalent in the same circle as $F(z)$. We remark that $p(z) \prec q(z)$ in Δ, q univalent, if and only if $p(rz) \prec q(rz)$, $0 < r < 1$. This implies, that for the proof of the theorem we can assume that $F(z)$ and, hence, $G(z)$ are convex univalent in Δ_{r0}, $r_0 > 1$.

Let $\phi(z) = G^{-1}(g(z))$. We see that $\phi(0) = 0$ and $\phi(z)$ is analytic in Δ_{r_1} where

$$r_1 = \min\{1, \sup\{r: g(\Delta_r) \cap \partial G(\Delta) = \emptyset\}\}.$$

Also $|\phi(z)| \leq 1$ in Δ_{r_1}. If $r_1 = 1$, the theorem is proven. Therefore let us assume $r_1 < 1$. $G(z)$ is univalent in Δ_{r0} and $g(z)$ analytic in Δ so we can conclude that $\phi(z)$ is analytic in $\overline{\Delta}_{r_1}$, and there exists a z_0, $|z_0| = r_1$, with $|\phi(z_0)| = 1$.

Furthermore we have

$$g(z) = G(\phi(z)) = \sum_{j=0}^{\infty} b_j \phi^j(z)$$

and the conditions $b_1 \neq 0$, $g'(0) = \cdots = g^{(n-1)}(0) = 0$ imply that $\phi'(0) = \cdots = \phi^{(n-1)}(0) = 0$. In $\overline{\Delta}_{r_1}$ we have $g'(z) = G'(\phi(z))\phi'(z)$, and for $z = z_0$, $\rho = z_0\phi'(z_0)/\phi(z_0) \geq n$ by Lemma 1.

We now find $g'(z)$ and $G'(z)$ by differentiating (2). These expressions involving $f(z)$ and $F(z)$ yield, after substituting the expression for ρ, the equation

$$g(z_0)(1 - n/\rho) + (n/\rho)f(z_0) = F(\phi(z_0)).$$

Since $\rho \geq n$, the left-hand side is a convex combination of $g(z_0)$ and $f(z_0)$ which are both interior points of the convex domain $F(\Delta)$ by Lemma 2 since $|z_0| < 1$. But $F(\phi(z_0))$ belongs to $\partial F(\Delta)$ and a contradiction follows.

Remark 1. In the notation of Theorem 1 we obviously have range $g(z) \subset$ range $G(z^n)$, and if $g(z) \neq 1$, $0 < |z| < 1$, it is easy to see that, in fact, $g(z) \prec G(z^n)$. In a private communication, D. Styer and D. Wright gave a counter-example to the latter formula if $g(z) - 1$ is allowed to have additional zeros.

Corollary 1. *Let* $f(z) = z + \sum_{j=n+1}^{\infty} a_j z^j \in K(\alpha)$, $1 - n/2 \leq \alpha < 1$. *Then*

$$\frac{f(z)}{z} \prec \frac{1}{z^{1/n}} \int_0^{z^{1/n}} \frac{d\tau}{(1 - \tau^n)^{(2-2\alpha)/n}} = G(z).$$

Proof. A straightforward computation shows that $g(z) = z(f'(z))^{1/(1-\alpha)}$ is starlike in Δ. It is clear that $g''(0) = \cdots = g^{(n)}(0) = 0$. Since $g(z)$ is starlike univalent in Δ, we may define an analytic function $w(z)$ in Δ by the equation

$$g(z)/z = 1/(1 - w(z))^{2/n}.$$

We now differentiate this equation and assume $|w(z_0)| = 1$ where $|z_0| < 1$.

Applying Lemma 1 we find after simplification that $\operatorname{Re}(zg'(z)/g(z)) = 1 - k/n \leq 0$ since $k \geq n$. This contradiction implies that $|w(z)| < 1$ in Δ and so $g(z)/z < 1/(1-z)^{2/n}$ or, equivalently, $f'(z) < 1/(1-z)^{\beta}$ where $\beta = (2-2\alpha)/n$. For $\beta \leq 1$, $1/(1-z)^{\beta}$ is convex univalent, and by Theorem 1 with $\gamma = 1$ the result follows.

Remark 2. If $n = 1$, Corollary 1 is exactly the subordination theorem on $K(\alpha)$, $\frac{1}{2} \leq \alpha < 1$ in [1, p. 427].

Corollary 2. *Let $f(z), g(z) \in K(0)$, $f''(0) = g''(0) = 0$. Then $f(z) + g(z)$ is starlike univalent in Δ.*

Proof. Recently D. Styer and D. Wright [8] developed a method to prove this result for any two functions f, g in $K(0)$ which satisfy

(i) $f(\Delta)$, $g(\Delta)$ contain the circle $\{w \mid |w| < \pi/4\}$,

(ii) $f(\Delta) \subset h(\Delta)$, $g(\Delta) \subset h(\Delta)$ where $h(z) = (2z)^{-1}\log(1+z)(1-z)^{-1}$.

Condition (i) follows from the well-known estimate $\arctan|z| \leq |p(z)|$ for every $p(z) \in K(0)$ with vanishing second coefficient. The second condition follows from Corollary 1 with $\alpha = 0$, $n = 2$ and by Remark 1.

Corollary 3. *Suppose $f(z) = z + a_2z^2 + \cdots$ and $F(z) = z + A_2z^2 + \cdots$ are analytic in Δ, $F'(z)$ is univalent and convex. If $f'(z) \prec F'(z)$, then $f(z)/z \prec F(z)/z$.*

Proof. This follows immediately from Theorem 1 by choosing $\gamma = 1$ and $n = 1$.

Remark 3. This completely generalizes a result of D. J. Hallenbeck [3] which corresponds to the choice $F'(z) = (1 + (1-2\alpha)z)/(1-z)$ where $0 \leq \alpha < 1$. An alternate proof of Corollary 3 follows from a general subordination result involving Hadamard convolutions in [6].

To conclude this paper we present a subordination theorem which is a simple but interesting corollary of the following lemma due to T. J. Suffridge [9].

Lemma 3. *Suppose that $f(z)$ is analytic in Δ, $F(z)$ is starlike univalent in Δ, and $f(z) \prec F(z)$. Then*

$$h(z) = \int_0^z \frac{f(\tau)}{\tau}\,d\tau \prec \int_0^z \frac{F(\tau)}{\tau}\,d\tau = H(z).$$

Theorem 2. *Suppose that $f(z) = z^p + a_{p+1}z^{p+1} + \cdots$ and $F(z) = z^p + A_{p+1}z^{p+1} + \cdots$ are analytic in Δ and that $g(z) = zf'(z)/f(z) \prec zF'(z)/F(z) = G(z)$. If $G(z)$ is univalent and starlike with respect to $w = p$, then $f(z)/z^p \prec F(z)/z^p$ where $p = 1, 2, \ldots$.*

Proof. We know that $f(z)/z^p = \exp h(z)$ and $F(z)/z^p = \exp H(z)$ where $h(z) = \int_0^z (g(\tau) - p)/\tau\,d\tau$ and $H(z) = \int_0^z (G(\tau) - p)/\tau\,d\tau$. The result now follows directly from Lemma 3.

Remark 4. This theorem generalizes the classical result of Strohhäcker [7] dealing with the class of starlike univalent functions which is achieved by taking $G(z) = (1+z)/(1-z)$. We note that among the interesting choices for $G(z)$ are the functions

$$p(1+(1-\alpha)\,z), \qquad p\left(\frac{1+\alpha z}{1-\alpha z}\right), \qquad p\left(\frac{1+(1-2\alpha)z}{1-z}\right)^{\beta},$$

where $p = 1, 2, 3, \ldots$, $0 \le \alpha < 1$ and $0 \le \beta \le 2$, and $(1+cz)/(1-z)$, where $|c| = 1$. These choices correspond to various compact families of starlike and spirallike mappings and, in the case $\beta > 1$ or $p \ge 2$, to nonunivalent mappings.

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