ON THE LOWER ORDER OF AN ENTIRE DIRICHLET SERIES

J. P. SINGH

ABSTRACT. The lower order $\lambda_s$ of $f(s)$ in each horizontal strip $S(a)$, with $a > A^*$, is equal to the lower order $\lambda$ of $f(s)$. The purpose of this note is to offer a proof of this result.

1. Let

$$f(s) = \sum_{n=1}^{\infty} a_n e^{-\lambda_n s}, \quad s = \sigma + it,$$

be a function represented by a Dirichlet series convergent in the whole plane where $\{\lambda_n\}^\infty_{n=1}$ is a sequence of positive, nondecreasing numbers with

$$\lim_{n \to \infty} \lambda_n + \lambda_{n+1} - \lambda_n > 0.$$ 

We shall use the following notations: For a fixed $t_0$, let $S(R)$ denote the horizontal strip $|t - t_0| < R$. Put

$$M_\sigma = \max_{-\infty < \sigma < \infty} |f(\sigma + it)|,$$ 

and let

$$\sup_{\sigma \to -\infty} \inf_{-\sigma} \log \log M_\sigma = \lambda, \quad \sup_{\sigma \to -\infty} \inf_{-\sigma} \log \log M_\sigma = \lambda_s.$$ 

Further, following Malliavin [2], we shall denote the maximum, upper and lower logarithmic densities of $\{\lambda_n\}$ by $\Delta^*$, $\Delta^0$ and $\Delta_0$ respectively.

2. Mandelbrojt and Gergen [3, pp. 219–220] have proven that the order $\rho_\sigma$ of $f(s)$ in each horizontal strip $S(a)$, with $a > D$, is equal to the order $\rho$ of $f(s)$. This result has been extended to the lower order $\lambda_s$ by Rahman [4]. But the proof of his theorem is not complete. Further, Rahman [5] improved the proof of his theorem under the additional hypothesis (satisfied

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if the coefficients are positive) that enables the original proof to work, but the additional hypothesis is unnatural.

In this note, our object is to prove a theorem which is better and more precise than the theorems of Rahman [4], [5], Roux [6] and Srivastava [7]. In the proof, we use Malliavin's version [2, p. 232] of Mandelbrojt's fundamental inequality. This gives a sharper result, since Malliavin's inequality involves a logarithmic density that is finer than the arithmetic density of Mandelbrojt's inequality. We prove the following.

**Theorem.** The lower order $\lambda_s$ of $f(s)$ in each horizontal strip $S(\pi a)$, with $a > 1$, is equal to the lower order $\lambda$ of $f(s)$.

3. **Proof.** It is known that there exists an increasing subsequence $\{n_j\}$ of $n$ for which

$$\limsup_{j \to \infty} \frac{\log |1/a_{n_j}|}{\lambda_{n_j} \log \lambda_{n_j-1}} = \frac{1}{\lambda} < \infty \quad [6].$$

Now, by Malliavin's version [2, p. 232] of Mandelbrojt's fundamental inequality, we get

$$\log M_s(\sigma_0) > -\left(1/\lambda + 2\epsilon\right)\log \lambda_{n_j} \log \lambda_{n_j-1} - \sigma_0 \lambda_{n_j} - \lambda_{n_j} \left[k(\lambda_{n_j}) - k \cdot (\lambda_{n_j})\right].$$

Since for sufficiently large $x$,

$$2(\Delta_0 - \epsilon) \log x < \lambda(x) < 2(\Delta_0^0 + \epsilon) \log x$$

and $k(x) = 2a \log x - \lambda(x)$, hence

$$2(a - \Delta_0 - \epsilon) \log x < k(x) < 2(a - \Delta_0^0 + \epsilon) \log x,$$

$$k \cdot (x) > 2(a - \Delta_0 - \epsilon) \log x.$$ We have

$$A = \limsup_{x \to \infty} \frac{k(x) - k \cdot (x)}{\log x} \leq 2(\Delta_0^0 - \Delta_0).$$

Under the hypothesis of the Theorem $\Delta_0^0 = \Delta_0$, therefore, we get from

$$\log M_s(\sigma_0) > -\left[\left(1/\lambda + 3\epsilon\right) \log \lambda_{n_j-1} + \sigma_0 \lambda_{n_j}\right].$$

Choose $\sigma_{j+1} = -(1/\lambda + 4\epsilon) \log \lambda_{n_j}$. For any $\sigma_0$ satisfying the inequalities $\sigma_{j+1} < \sigma_0 \leq \sigma_j$, $M_s(\sigma_0)$ is decreasing for increasing $\sigma_0$. Hence, we have

$$\lambda_s = \liminf_{\sigma_0 \to -\infty} \frac{\log \log M_s(\sigma_0)}{-\sigma_0} \geq \liminf_{j \to \infty} \frac{(1 + o(1)) \log \lambda_{n_j}}{(1/\lambda + 4\epsilon) \log \lambda_{n_j}} = \frac{1}{(1/\lambda + 4\epsilon)}.$$ \[1\] Notations used here are same as Malliavin's [2].
Since $\epsilon$ is arbitrary, $\lambda_s \geq \lambda$. But $\lambda_s \leq \lambda$ always. The case $\lambda = 0$ is obvious. This leads to the desired conclusion.

4. Remarks. 1. The errors and omissions in the proof of Rahman's theorem were pointed out by Sungar i Balaguer [8], but he did not provide the proof of Theorem 1 of [4].

2. Our definition of $\lambda_s$ is slightly different from those used in [4], [5] and [7].

3. Roux [6] has used a different definition of lower order in the strip (see Blambert [1]).

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DEPARTMENT OF MATHEMATICS, KURUKSHETRA UNIVERSITY, KURUKSHETRA-132119 INDIA