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A FACTORIZATION THEOREM IN $H^1(U^3)$

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ABSTRACT. It is shown that there exists $f \in H^1(U^3)$ which is not the product of two functions in $H^2(U^3)$. This partially answers a question posed by W. Rudin.

We let U be the unit disc in the complex plane, T the unit circle, U^n and T^n the Cartesian products of n copies of U and T respectively, m_n normalized Lebesgue measure on T^n, and $H^p(U^n)$ the usual Hardy class of holomorphic functions on U^n.

It is well known that every $f \in H^1(U)$ can be expressed as the product of functions g and h in $H^2(U)$. In fact if B denotes the Blaschke product of the zeros of f, it is sufficient to let $g = \sqrt{f/B}$ and $h = Bg$. Rudin [1], [2, p. 63] has shown that if $n \geq 4$, then not every $f \in H^1(U^n)$ can be factored into a product of functions in $H^2(U^n)$. It is the purpose of this note to show that Rudin's method may be extended to cover the three-dimensional case as well.

Theorem. *There exists $f \in H^1(U^3)$ which is not expressible as a product of two functions in $H^2(U^3)$.*

Thus we observe that for this factorization problem only the case $n = 2$ remains open. That case does not seem to be accessible by these techniques. (See the last paragraph of this paper.)

We begin with two lemmas.

Lemma 1. *The ratio $\|(z+w)^{2q}\|_1/\|(z+w)^{2q}\|_2$ tends to zero as the integer q tends to infinity.*

Proof. Since

$$(z+w)^q = \sum_{k=0}^{q} \binom{q}{k} z^k w^{q-k},$$

we have

$$\|(z+w)^{2q}\|_1 = \|(z+w)^q\|_2^2 = \sum_{k=0}^{q} \binom{q}{k}^2 = \binom{2q}{q}.$$

Similarly

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[2] The author is indebted to W. Rudin for suggesting an improvement in the original form of Lemma 2.

$$\|(z+w)^{2q}\|_2^2 = \sum_{k=0}^{2q} \binom{2q}{k}^2 = \binom{4q}{2q}.$$

An application of Stirling's formula yields

$$\frac{\|(z+w)^{2q}\|_1}{\|(z+w)^{2q}\|_2} = \frac{((2q)!)^2}{(q!)^2((4q)!)^{1/2}} \sim \left(\frac{2}{\pi q}\right)^{1/4},$$

proving the lemma.

Lemma 2. *Suppose $\epsilon \neq 0$ and $Q(z, w)$ is a homogeneous polynomial of degree $N-1$ in which the monomial w^{N-1} appears with a nonzero coefficient. If g and h are holomorphic functions on U^2 such that the expansion of gh in homogeneous polynomials has the form*

$$g(z, w)h(z, w) = \epsilon z^N + w^2 Q(z, w) + F_{N+2} + F_{N+3} + \cdots,$$

where F_j is a homogeneous polynomial of degree j, then $\|g\|_2\|h\|_2 \geq \|Q\|_2$.

Proof. Suppose $g = \Sigma_{j=k}^{\infty} G_j$ and $h = \Sigma_{j=N-k}^{\infty} H_j$ are the expansions of g and h in homogeneous polynomials G_j and H_j of degree j with $G_k \not\equiv 0$ and $H_{N-k} \not\equiv 0$. We have

$$G_k H_{N-k} = \epsilon z^N \tag{1}$$

and

$$G_k H_{N+1-k} + G_{k+1} H_{N-k} = w^2 Q(z, w). \tag{2}$$

From (1) we deduce that G_k and H_{N-k} are both simply powers of z. If $0 < k < N$ it follows that z divides the left side of (2) but not the right side. Thus without loss of generality we assume that $k = 0$ and in fact that $G_0 = 1$ and $H_N = \epsilon z^N$. From (2) we conclude

$$H_{N+1} + G_1 H_N = w^2 Q(z, w).$$

Now $G_1 H_N$ is of the form $\alpha z^{N+1} + \beta z^N w$ and consequently is orthogonal to $w^2 Q(z, w)$ in $H^2(U^2)$. Thus

$$\|g\|_2^2 \|h\|_2^2 \geq |G_0|^2 \|H_{N+1}\|_2^2 = \|H_{N+1}\|_2^2 = \|w^2 Q(z, w) - G_1 H_N\|_2^2$$

$$\geq \|w^2 Q(z, w)\|_2^2 = \|Q(z, w)\|_2^2.$$

This proves Lemma 2.

We note from Lemma 1 that for any positive integer n there exist positive numbers R_n and ϵ_n, $0 < \epsilon_n < 1/2$, and a positive integer N_n such that

$$\|\epsilon_n z^{N_n} + R_n w^2 (z+w)^{N_n - 1}\|_1 < 1/n \quad \text{and} \quad \|R_n w^2 (z+w)^{N_n - 1}\|_2 > n^2.$$

Without loss of generality we may take $N_{n+1} \geq 2 + N_n$. We define

$$P_n(z, w) = \epsilon_n z^{N_n} + R_n w^2 (z+w)^{N_n - 1}.$$

To prove the theorem we must produce a function $f \in H^1(U^3)$ which is not the product of two functions in $H^2(U^3)$. Our construction from this point is finished by a repetition of the second portion of Rudin's construction in the four-dimensional case. For the convenience of the reader we provide the outline of that argument.

Lemma 2 implies the existence for each n of $\delta_n > 0$ such that if

$$(3) \qquad g(z, w)h(z, w) = F_0 + \cdots + F_{N_n - 1} + P_n(z, w) + F_{N_n + 2} + \cdots,$$

where F_j is a homogeneous polynomial of degree j with $\|F_j\|_1 < \delta_n$ for $0 \le j \le N_n - 1$, then

$$(4) \qquad \|g\|_2 \|h\|_2 \ge n^2.$$

Let I_n be disjoint open arcs on T with $m_1(I_n) = n^{-2}\delta_n$. Let $\psi_n \in H^\infty(U)$ be such that $|\psi_n^*| = \delta_n^{-1}$ on I_n and $|\psi_n^*| = n^{-2}$ off $\overline{I}_n$. (Here ψ_n^* denotes the radial limit of ψ_n.) Finally, let

$$f(z, w, t) = \sum_{n=1}^{\infty} P_n(z, w)\psi_n(t).$$

From the fact that $\|P_n\|_1 < n^{-1}$ in $H^1(U^2)$ and $\|\psi_n\|_1 < 2n^{-2}$ in $H^1(U)$, it follows easily that $f \in H^1(U^3)$.

Suppose $f = gh$. For a fixed $t \in I_n$ we have $|\psi_j^*(t)/\psi_n^*(t)| \le \delta_n$ for $j \ne n$. By fixing $t \in I_n$ and considering

$$\psi_n^*(t)^{-1} f(z, w, t) = P_n(z, w) + \sum_{j \ne n} (\psi_j^*(t)/\psi_n^*(t)) P_j(z, w),$$

we conclude from (3) and (4) that

$$\delta_n \left\{ \int_{T^2} |g^*(z, w, t)|^2 \, dm_2(z, w) \right\}^{1/2} \left\{ \int_{T^2} |h^*(z, w, t)|^2 \, dm_2(z, w) \right\}^{1/2} \ge n^2.$$

We then apply the inequality relating the geometric and arithmetic means, integrate the result over I_n, and sum on n to obtain $\|g\|_2^2 + \|h\|_2^2 = \infty$, giving the desired conclusion.

It is perhaps worth noting that Lemmas 1 and 2 show that the mapping $\mu_2 \colon H^2(U^2) \times H^2(U^2) \to H^1(U^2)$, given by $\mu_2(g, h) = gh$, is not open at the origin. The quantities P_n of our proof have the property that $\|P_n\|_1 \to 0$ as $n \to \infty$, but for no value of n is P_n in the image of the product of the unit ball with itself. It is well known that the corresponding mapping in the one-dimensional case is open at the origin.

Remark. After this manuscript was submitted, J. P. Rosay showed μ_2 is not onto by using the result in the paragraph above. The factorization problem is thus settled in all dimensions. Rosay's paper will appear in the Illinois Journal of Mathematics.

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