QUASI-NILPOTENT SETS IN SEMIGROUPS

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ABSTRACT. In a compact semigroup $S$ with zero $0$, a subset $A$ of $S$ is called quasi-nilpotent if the closed semigroup generated by $A$ contains $0$. A probability measure $\mu$ on $S$ is called nilpotent if the sequence $(\mu^n)$ converges to the Dirac measure at $0$. It is shown that a probability measure is nilpotent if and only if its support is quasi-nilpotent. Consequently, the set of all nilpotent measures on $S$ is convex and everywhere dense in the set of all probability measures on $S$ and the union of their supports is $S$.

In a topological semigroup with zero $0$, an element $x$ is termed nilpotent if $x^n \to 0$ as $n \to \infty$ [5]. This definition has an obvious extension to subsets of the semigroup, i.e. a subset $A$ is nilpotent if $A^n \to 0$ as $n \to \infty$. Now we call a subset $B$ of the semigroup quasi-nilpotent if the closed semigroup generated by $B$ contains the zero $0$. It is shown that, when the topological semigroup is compact, a singleton is nilpotent if and only if quasi-nilpotent. Then we investigate the set of probability measures on a compact semigroup and characterize a nilpotent probability measure as a measure with quasi-nilpotent support.

Let $S$ be a topological semigroup with zero $0$, and $A$ a subset of $S$. Let $S(A)$ denote the semigroup generated by $A$, i.e. $S(A) = \bigcup_{n=1}^{\infty} A^n$. It is trivial that any subset containing $0$ is quasi-nilpotent; in particular, the set $N_S$ of nilpotent elements of $S$ is quasi-nilpotent. From the semigroup $S$ given in Example 6 below, in which $N_S = [0, 1)$ and $N_S^n = N_S$ for all $n$ [4, p. 56], we see that $N_S$ is not nilpotent.

Theorem 1. Let $A$ be a subset of $S$. Then (i) if $S(A) \cap N_S = \emptyset$ (where the bar denotes closure), then $A$ is quasi-nilpotent.

(ii) If $A^n$ is quasi-nilpotent for some $n$, then $A$ itself is quasi-nilpotent.

Proof. (i) Take $a \in S(A) \cap N_S$. In view of the fact that $a^n \to 0$, we have $0 \in S(A)$, i.e. $A$ is quasi-nilpotent.

(ii) Since $S(A^n) \subset S(A)$ and $0 \in S(A^n)$, it follows that $0 \in S(A)$, and the theorem is proved.

We remark that, if $A^n$ is nilpotent for some $n$, then $A$ is also nilpotent, by a similar argument to that given in the proof of Lemma 2.1.4 of [4].
Evidently a nilpotent set is quasi-nilpotent. As for the converse, which may not be true in general, we prove a special case in

Theorem 2. Suppose $S$ is a compact semigroup with $0$. Then $x \in S$ is nilpotent if and only if quasi-nilpotent.

Proof. It is enough to show that $x$ is nilpotent if it is quasi-nilpotent. Recall that the minimal ideal $K(S(x))$ of the compact semigroup $S(x)$ contains exactly all cluster points of the sequence $(x^n)_{n=1}^{\infty}$ (see, for example, [4, Theorem 3.1.1]). Now $K(S(x)) = \{0\}$ since $0 \in S(x)$. Thus the sequence $(x^n)$ has a unique cluster point, whence $x^n \to 0$ as $n \to \infty$, completing the proof.

Remark. The preceding theorem does not hold for a compact semitopological semigroup (i.e. the multiplication is only separately continuous). For instance, take the compact monothetic semigroup $S(\mu)$ generated by $\mu$, with $\mu$ defined in Example 2 of [1]; then the semigroup has zero $0$ and identity $1$ such that $u^{n!/2} \to 0$ and $u^{n!} \to 1$. As a consequence, the element $u$ is quasi-nilpotent but not nilpotent.

In what follows $S$ will be a compact semigroup with zero $0$. Denote by $P(S)$ the set of probability measures (i.e. normalized positive regular Borel measures) on $S$. For $\mu, \nu \in P(S)$, define convolution $\mu \ast \nu \in P(S)$ by

$$\int f(z) \, d(\mu \ast \nu)(z) = \int \int f(xy) \, d\mu(x) \, d\nu(y)$$

for all continuous functions $f$ on $S$, so that $P(S)$ forms a semigroup. If $P(S)$ is endowed with the weak* topology, i.e. a net $(\mu_\alpha)$ in $P(S)$ converges to $\mu \in P(S)$ if $f(x) \, d\mu_\alpha(x) \to f(x) \, d\mu(x)$ for continuous functions $f$ on $S$, then $P(S)$ is a compact semigroup [3].

The support of $\mu \in P(S)$, $\text{supp } \mu$, is the smallest closed set with $\mu$-mass $1$. It is well known [3, Lemma 2.1] that, for $\mu, \nu \in P(S)$, $\text{supp } (\mu \ast \nu) = (\text{supp } \mu) \cdot (\text{supp } \nu)$.

Let $\Gamma$ be a subset of $P(S)$ and define its support as the set $\text{supp } \Gamma = \bigvee_{\mu \in \Gamma} \text{supp } \mu$. It is easy to see that $\text{supp } (\Gamma_1 \cap \Gamma_2) = (\text{supp } \Gamma_1) \cap (\text{supp } \Gamma_2)$ for $\Gamma_1 \subseteq P(S), \Gamma_2 \subseteq P(S)$.

Lemma 3. Let $\Gamma \subseteq P(S)$. Then $\text{supp } S(\Gamma) = \text{supp } S(\Gamma) = \overline{S(\text{supp } \Gamma)}$.

Proof. That $\overline{S(\Gamma)} = \text{supp } S(\Gamma)$ follows from a result in [3, p. 55]. We assert that $\text{supp } S(\Gamma) = \overline{S(\text{supp } \Gamma)}$. Since $S(\Gamma) \supseteq \Gamma^n$ for $n = 1, 2, \ldots$, clearly $S(\Gamma) \supseteq \text{supp } \Gamma^n = \overline{S(\text{supp } \Gamma)^n}$ and so $S(\Gamma) \supseteq S(\text{supp } \Gamma)$. Whence $\text{supp } S(\Gamma) \supseteq S(\text{supp } \Gamma)$. On the other hand, take any $\mu \in S(\Gamma)$. Then $\mu \in \Gamma^n$ for some $n$, implying that $\text{supp } \mu \subseteq \text{supp } \Gamma^n = \overline{S(\text{supp } \Gamma)^n} \subseteq S(\text{supp } \Gamma)$. This gives $\text{supp } S(\Gamma) \subseteq \overline{S(\text{supp } \Gamma)}$, and the result follows.

Since the Dirac measure $\theta$ at $0$ is a zero in $P(S)$, we can now consider quasi-nilpotent sets in $P(S)$. 

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Theorem 4. A subset $\Gamma \subset P(S)$ is quasi-nilpotent if and only if $\text{supp} \, \Gamma$ is quasi-nilpotent in $S$.

Proof. Suppose first that $\Gamma$ is quasi-nilpotent, i.e. $\theta \in \overline{S(\Gamma)}$. By virtue of Lemma 3, we have $0 \in \overline{S(\text{supp} \, \Gamma)}$, i.e. $\text{supp} \, \Gamma$ is quasi-nilpotent. Conversely, suppose $\text{supp} \, \Gamma$ is quasi-nilpotent in $S$. This means that $0 \in \overline{S(\text{supp} \, \Gamma)}$ and therefore $|0|$ is the minimal ideal $K(\overline{S(\text{supp} \, \Gamma)})$ of the semigroup $\overline{S(\text{supp} \, \Gamma)}$.

Now consider the minimal ideal $K(\overline{S(\Gamma)})$ of the compact semigroup $\overline{S(\Gamma)}$ [6, Theorem 2]. Since $\overline{S(\text{supp} \, \Gamma)} = \overline{S(\text{supp} \, \Gamma)}$ (see, for example, [2, Theorem 5(2)]) and $\overline{S(\Gamma)} = \overline{S(\text{supp} \, \Gamma)}$ by Lemma 3, we have $|0| = \text{supp} \, K(\overline{S(\Gamma)})$, giving that $K(\overline{S(\Gamma)}) = \{\theta\}$. Accordingly $\theta \in \overline{S(\Gamma)}$, and the theorem is proved.

By Theorems 2 and 4, we immediately obtain

Theorem 5. A measure $\mu \in P(S)$ is nilpotent if and only if $\text{supp} \, \mu$ is quasi-nilpotent in $S$.

Example 6. The result in Theorem 5 is best possible in the sense that the support of a nilpotent measure in $P(S)$ need not be a nilpotent subset of $S$. Take the semigroup $S = [0, 1]$ with the usual topology and the ordinary multiplication. Let $\mu$ be the restriction to $S$ of the Lebesgue measure on the real line. Since $\text{supp} \, \mu = S$ is quasi-nilpotent, it follows that $\mu$ is nilpotent. However, $\text{supp} \, \mu$ is not nilpotent since $(\text{supp} \, \mu)^n = \text{supp} \, \mu = S$ for all $n$.

Note that Theorem 5 is not true for the compact semitopological semigroup $S_{\mu}(\mu)$ considered in the Remark above. Obviously the Dirac measure $\delta(u)$ at $u$ is not nilpotent while $\text{supp} \, \delta(u)$ is quasi-nilpotent in $S$.

Applying Theorem 5, we obtain the following results about the set $N(P(S))$ of nilpotent elements in $P(S)$. First we have a sufficient condition for a probability measure to be nilpotent.

Theorem 7. Let $\mu \in P(S)$. If $\text{supp} \, \mu \cap N(S) \neq \emptyset$, then $\mu \in N(P(S))$.

Proof. Since $\overline{S(\text{supp} \, \mu)} \cap N(S) \supset \text{supp} \, \mu \cap N(S) \neq \emptyset$, we see that the set $\text{supp} \, \mu$ is quasi-nilpotent in $S$ by Theorem 1 (i). Whence $\mu$ is nilpotent.

Example 8. The converse of Theorem 7 may not hold. For instance, take the semigroup $S$ with the following multiplication table:

$$
\begin{array}{ccc}
0 & a & b & c \\
0 & 0 & 0 & 0 \\
a & 0 & 0 & a \\
b & 0 & 0 & b \\
c & 0 & a & a & c \\
\end{array}
$$

Then the measure $\mu = \frac{1}{2} \delta(b) + \delta(c) \in N(P(S))$ since $0 \in \text{supp} \, \mu^2$. However, $\text{supp} \, \mu \cap N(S) = \{b, c\} \cap \{0, a\} = \emptyset$. 

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Corollary 9. (i) $N(P(S))$ is a noncountable set.
(ii) $\bigcup \{\text{supp } \mu : \mu \in N(P(S))\} = S$.

Proof. (i) Take any measure $\mu \neq \theta$ and real number $0 < t < 1$. Then the measure $t \mu + (1 - t)\theta$ is nilpotent since $0 \in \text{supp } (t \mu + (1 - t)\theta) \cap N(S)$. Hence the set $N(P(S)) \supset \{t \mu + (1 - t)\theta : 0 < t < 1\}$ and so is noncountable.

(ii) Let $a \in S$. Since $0 \in \text{supp } \frac{1}{2}(\delta(a) + \theta) \cap N(S)$, it follows that $\frac{1}{2}(\delta(a) + \theta) \in N(P(S))$. That $a \in \text{supp } \frac{1}{2}(\delta(a) + \theta)$ gives the result.

A semigroup with zero is said to be nil if each element is nilpotent.

Theorem 10. $P(S)$ is nil if and only if $S$ is nil.

Proof. The "if" part follows from the fact that, for $\mu \in P(S)$, $\text{supp } \mu \cap N(S) = \text{supp } \mu \neq \emptyset$. To prove the "only if" part, take $a \in S$ and note that $\delta(a)$ is nilpotent in $P(S)$. So $a$ is nilpotent in $S$ and the proof is complete.

Lemma 11. Let $\mu, \nu \in P(S)$. If $\mu \in N(P(S))$ and $\text{supp } \mu \subset \text{supp } \nu$, then $\nu \in N(P(S))$.

Proof. This is immediate since $0 \in \overline{S(\text{supp } \mu)} \subset \overline{S(\text{supp } \nu)}$.

Theorem 12. (i) $N(P(S))$ is a convex set and hence connected.
(ii) $N(P(S)) = P(S)$.

Proof. (i) Take $\mu, \nu \in N(P(S))$. For real number $0 < t < 1$, the measure $t \mu + (1 - t)\nu \in N(P(S))$ since
\[
\text{supp } (t \mu + (1 - t)\nu) = \text{supp } \mu \cup \text{supp } \nu \supset \text{supp } \mu.
\]
Thus $N(P(S))$ is convex.

(ii) Let $\tau \in P(S)$. Clearly $\theta/n + (n - 1)\tau/n \in N(P(S))$ for any positive integer $n$. As the sequence $(\theta/n + (n - 1)\tau/n)_{n=1}^{\infty}$ converges to $\tau$, we see that $N(P(S))$ is dense in $P(S)$.

Corollary 13. Let $W$ be a subset of $P(S)$. If $W \supset N(P(S))$, then $W$ is a connected set.

Proof. This follows simply from the previous theorem.

For any $\mu \in P(S)$, it is a well-known fact that the sequence $((\mu + \mu^2 + \cdots + \mu^n)/n)_{n=1}^{\infty}$ must converge to a measure $L(\mu) \in P(S)$ such that $\text{supp } L(\mu)$ is the minimal ideal of the semigroup $\overline{S(\text{supp } \mu)}$; see [7] or [8].

Theorem 14. The measure $\mu \in P(S)$ is nilpotent if and only if $L(\mu) = \theta$.

Proof. In view of the fact that $L(\mu) = \theta$ if and only if $\overline{S(\text{supp } \mu)}$ contains $0$, we apply Theorem 5 to conclude the proof.
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REFERENCES


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