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QUASI-NILPOTENT SETS IN SEMIGROUPS

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ABSTRACT. In a compact semigroup S with zero 0, a subset A of S is called quasi-nilpotent if the closed semigroup generated by A contains 0. A probability measure μ on S is called nilpotent if the sequence (μ^n) converges to the Dirac measure at 0. It is shown that a probability measure is nilpotent if and only if its support is quasi-nilpotent. Consequently, the set of all nilpotent measures on S is convex and everywhere dense in the set of all probability measures on S and the union of their supports is S.

In a topological semigroup with zero 0, an element x is termed nilpotent if $x^n \longrightarrow 0$ as $n \longrightarrow \infty$ [5]. This definition has an obvious extension to subsets of the semigroup, i.e. a subset A is nilpotent if $A^n \longrightarrow 0$ as $n \longrightarrow \infty$. Now we call a subset B of the semigroup *quasi-nilpotent* if the closed semigroup generated by B contains the zero 0. It is shown that, when the topological semigroup is compact, a singleton is nilpotent if and only if quasi-nilpotent. Then we investigate the set of probability measures on a compact semigroup and characterize a nilpotent probability measure as a measure with quasi-nilpotent support.

Let S be a topological semigroup with zero 0, and A a subset of S. Let $S(A)$ denote the semigroup generated by A, i.e. $S(A) = \bigcup_{n=1}^{\infty} A^n$. It is trivial that any subset containing 0 is quasi-nilpotent; in particular, the set $N(S)$ of nilpotent elements of S is quasi-nilpotent. From the semigroup S given in Example 6 below, in which $N(S) = [0, 1)$ and $N(S)^n = N(S)$ for all n [4, p. 56], we see that $N(S)$ is not nilpotent.

Theorem 1. *Let A be a subset of S. Then* (i) *If $\overline{S(A)} \cap N(S) \neq \emptyset$ (where the bar denotes closure), then A is quasi-nilpotent.*

(ii) *If A^n is quasi-nilpotent for some n, then A itself is quasi-nilpotent.*

Proof. (i) Take $a \in \overline{S(A)} \cap N(S)$. In view of the fact that $a^n \longrightarrow 0$, we have $0 \in \overline{S(A)}$, i.e. A is quasi-nilpotent.

(ii) Since $S(A^n) \subset S(A)$ and $0 \in \overline{S(A^n)}$, it follows that $0 \in \overline{S(A)}$, and the theorem is proved.

We remark that, if A^n is nilpotent for some n, then A is also nilpotent, by a similar argument to that given in the proof of Lemma 2.1.4 of [4].

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Evidently a nilpotent set is quasi-nilpotent. As for the converse, which may not be true in general, we prove a special case in

Theorem 2. *Suppose* S *is a compact semigroup with* 0. *Then* $x \in S$ *is nilpotent if and only if quasi-nilpotent.*

Proof. It is enough to show that x is nilpotent if it is quasi-nilpotent. Recall that the minimal ideal $K(\overline{S(x)})$ of the compact semigroup $\overline{S(x)}$ contains exactly all cluster points of the sequence $(x^n)_{n=1}^{\infty}$ (see, for example, [4, Theorem 3.1.1]). Now $K(\overline{S(x)}) = \{0\}$ since $0 \in \overline{S(x)}$. Thus the sequence (x^n) has a unique cluster point, whence $x^n \to 0$ as $n \to \infty$, completing the proof.

Remark. The preceding theorem does not hold for a compact semitopological semigroup (i.e. the multiplication is only separately continuous). For instance, take the compact monothetic semigroup $S_w(\mu)$ generated by u, with μ defined in Example 2 of [1]; then the semigroup has zero 0 and identity 1 such that $u^{n!/2} \to 0$ and $u^{n!} \to 1$. As a consequence, the element u is quasi-nilpotent but not nilpotent.

In what follows S will be a compact semigroup with zero 0. Denote by $P(S)$ the set of probability measures (i.e. normalized positive regular Borel measures) on S. For $\mu, \nu \in P(S)$, define convolution $\mu\nu \in P(S)$ by

$$\int f(z)\, d\mu\nu(z) = \iint f(xy)\, d\mu(x)\, d\nu(y)$$

for all continuous functions f on S, so that $P(S)$ forms a semigroup. If $P(S)$ is endowed with the weak* topology, i.e. a net (μ_α) in $P(S)$ converges to $\mu \in P(S)$ if $\int f(x)\, d\mu_\alpha(x) \to \int f(x) d\mu(x)$ for continuous functions f on S, then $P(S)$ is a compact semigroup [3].

The support of $\mu \in P(S)$, supp μ, is the smallest closed set with μ-mass 1. It is well known [3, Lemma 2.1] that, for $\mu, \nu \in P(S)$, supp $(\mu\nu) = (\text{supp } \mu) \cdot (\text{supp } \nu)$.

Let Γ be a subset of $P(S)$ and define its support as the set supp $\Gamma = \overline{\bigcup_{\mu \in \Gamma} \text{supp } \mu}$. It is easy to see that $\text{supp}(\Gamma_1 \Gamma_2) = (\text{supp } \Gamma_1)(\text{supp } \Gamma_2)$ for $\Gamma_1 \subset P(S)$, $\Gamma_2 \subset P(S)$.

Lemma 3. *Let* $\Gamma \subset P(S)$. *Then* $\text{supp } \overline{S(\Gamma)} = \text{supp } S(\Gamma) = \overline{S(\text{supp } \Gamma)}$.

Proof. That $\text{supp } \overline{S(\Gamma)} = \text{supp } S(\Gamma)$ follows from a result in [3, p. 55]. We assert that $\text{supp } S(\Gamma) = \overline{S(\text{supp } \Gamma)}$. Since $S(\Gamma) \supset \Gamma^n$ for $n = 1, 2, \ldots$, clearly $\text{supp } S(\Gamma) \supset \text{supp } \Gamma^n = (\text{supp } \Gamma)^n$ and so $\text{supp } S(\Gamma) \supset S(\text{supp } \Gamma)$. Whence $\text{supp } S(\Gamma) \supset \overline{S(\text{supp } \Gamma)}$. On the other hand, take any $\mu \in S(\Gamma)$. Then $\mu \in \Gamma^n$ for some n, implying that $\text{supp } \mu \subset \text{supp } \Gamma^n = (\text{supp } \Gamma)^n \subset S(\text{supp } \Gamma) \subset \overline{S(\text{supp } \Gamma)}$. This gives $\text{supp } S(\Gamma) \subset \overline{S(\text{supp } \Gamma)}$, and the result follows.

Since the Dirac measure θ at 0 is a zero in $P(S)$, we can now consider quasi-nilpotent sets in $P(S)$.

Theorem 4. *A subset* $\Gamma \subset P(S)$ *is quasi-nilpotent if and only if* supp Γ *is quasi-nilpotent in* S.

Proof. Suppose first that Γ is quasi-nilpotent, i.e. $\theta \in \overline{S(\Gamma)}$. By virtue of Lemma 3, we have $0 \in \overline{S(\text{supp }\Gamma)}$ i.e. supp Γ is quasi-nilpotent. Conversely, suppose supp Γ is quasi-nilpotent in S. This means that $0 \in \overline{S(\text{supp }\Gamma)}$ and therefore $\{0\}$ is the minimal ideal $K(\overline{S(\text{supp }\Gamma)})$ of the semigroup $\overline{S(\text{supp }\Gamma)}$. Now consider the minimal ideal $K(\overline{S(\Gamma)})$ of the compact semigroup $\overline{S(\Gamma)}$ [6, Theorem 2]. Since supp $K(\overline{S(\Gamma)}) = K(\text{supp }\overline{S(\Gamma)})$ (see, for example, [2, Theorem 5(2)]) and supp $\overline{S(\Gamma)} = \overline{S(\text{supp }\Gamma)}$ by Lemma 3, we have $\{0\} =$ supp $K(\overline{S(\Gamma)})$, giving that $K(\overline{S(\Gamma)}) = \{\theta\}$. Accordingly $\theta \in \overline{S(\Gamma)}$, and the theorem is proved.

By Theorems 2 and 4, we immediately obtain

Theorem 5. *A measure* $\mu \in P(S)$ *is nilpotent if and only if* supp μ *is quasi-nilpotent in* S.

Example 6. The result in Theorem 5 is best possible in the sense that the support of a nilpotent measure in $P(S)$ need not be a nilpotent subset of S. Take the semigroup $S = [0, 1]$ with the usual topology and the ordinary multiplication. Let μ be the restriction to S of the Lebesgue measure on the real line. Since supp $\mu = S$ is quasi-nilpotent, it follows that μ is nilpotent. However, supp μ is not nilpotent since $(\text{supp }\mu)^n = \text{supp }\mu = S$ for all n.

Note that Theorem 5 is not true for the compact semitopological semigroup $S_w(\mu)$ considered in the Remark above. Obviously the Dirac measure $\delta(u)$ at u is not nilpotent while supp $\delta(u)$ is quasi-nilpotent in S.

Applying Theorem 5, we obtain the following results about the set $N(P(S))$ of nilpotent elements in $P(S)$. First we have a sufficient condition for a probability measure to be nilpotent.

Theorem 7. *Let* $\mu \in P(S)$. *If* supp $\mu \cap N(S) \neq \varnothing$, *then* $\mu \in N(P(S))$.

Proof. Since $\overline{S(\text{supp }\mu)} \cap N(S) \supset \text{supp }\mu \cap N(S) \neq \varnothing$, we see that the set supp μ is quasi-nilpotent in S by Theorem 1 (i). Whence μ is nilpotent.

Example 8. The converse of Theorem 7 may not hold. For instance, take the semigroup S with the following multiplication table:

	0	a	b	c
0	0	0	0	0
a	0	0	a	0
b	0	0	b	0
c	0	a	a	c

Then the measure $\mu = \frac{1}{2}(\delta(b) + \delta(c)) \in N(P(S))$ since $0 \in \text{supp }\mu^2$. However, supp $\mu \cap N(S) = \{b, c\} \cap \{0, a\} = \varnothing$.

Corollary 9. (i) *$N(P(S))$ is a noncountable set.*

(ii) $\bigcup\{\text{supp } \mu : \mu \in N(P(S))\} = S$.

Proof. (i) Take any measure $\mu \neq \theta$ and real number $0 \leq t < 1$. Then the measure $t\mu + (1-t)\theta$ is nilpotent since $0 \in \text{supp } (t\mu + (1-t)\theta) \cap N(S)$. Hence the set $N(P(S)) \supset \{t\mu + (1-t)\theta : 0 \leq t < 1\}$ and so is noncountable.

(ii) Let $a \in S$. Since $0 \in \text{supp } \frac{1}{2}(\delta(a) + \theta) \cap N(S)$, it follows that $\frac{1}{2}(\delta(a) + \theta) \in N(P(S))$. That $a \in \text{supp } \frac{1}{2}(\delta(a) + \theta)$ gives the result.

A semigroup with zero is said to be nil if each element is nilpotent.

Theorem 10. *$P(S)$ is nil if and only if S is nil.*

Proof. The "if" part follows from the fact that, for $\mu \in P(S)$, $\text{supp } \mu \cap N(S) = \text{supp } \mu \neq \emptyset$. To prove the "only if" part, take $a \in S$ and note that $\delta(a)$ is nilpotent in $P(S)$. So a is nilpotent in S and the proof is complete.

Lemma 11. *Let $\mu, \nu \in P(S)$. If $\mu \in N(P(S))$ and $\text{supp } \mu \subset \text{supp } \nu$, then $\nu \in N(P(S))$.*

Proof. This is immediate since $0 \in \overline{S(\text{supp } \mu)} \subset \overline{S(\text{supp } \nu)}$.

Theorem 12. (i) *$N(P(S))$ is a convex set and hence connected.*

(ii) $\overline{N(P(S))} = P(S)$.

Proof. (i) Take $\mu, \nu \in N(P(S))$. For real number $0 < t < 1$, the measure $t\mu + (1-t)\nu \in N(P(S))$ since

$$\text{supp } (t\mu + (1-t)\nu) = \text{supp } \mu \cup \text{supp } \nu \supset \text{supp } \mu.$$

Thus $N(P(S))$ is convex.

(ii) Let $\tau \in P(S)$. Clearly $\theta/n + (n-1)\tau/n \in N(P(S))$ for any positive integer n. As the sequence $(\theta/n + (n-1)\tau/n)_{n=1}^{\infty}$ converges to τ, we see that $N(P(S))$ is dense in $P(S)$.

Corollary 13. *Let W be a subset of $P(S)$. If $W \supset N(P(S))$, then W is a connected set.*

Proof. This follows simply from the previous theorem.

For any $\mu \in P(S)$, it is a well-known fact that the sequence $((\mu + \mu^2 + \cdots + \mu^n)/n)_{n=1}^{\infty}$ must converge to a measure $L(\mu) \in P(S)$ such that $\text{supp } L(\mu)$ is the minimal ideal of the semigroup $\overline{S(\text{supp } \mu)}$; see [7] or [8].

Theorem 14. *The measure $\mu \in P(S)$ is nilpotent if and only if $L(\mu) = \theta$.*

Proof. In view of the fact that $L(\mu) = \theta$ if and only if $\overline{S(\text{supp } \mu)}$ contains 0, we apply Theorem 5 to conclude the proof.

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