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QUADRATIC FORMS IN HARMONIC ANALYSIS AND THE BOCHNER-EBERLEIN THEOREM[1]

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ABSTRACT. New characterizations are given of the classes of functions on a locally compact Abelian group which are Fourier-Stieltjes transforms of: bounded measures, nonnegative bounded measures, and integrable functions. These characterizations are all of the same form, *viz.*, that a positive definite Hermitian form dominate a symmetric quadratic form.

1. **Introduction.** If G is a locally compact Abelian group with dual group $\hat{G}$, we denote by $M(\hat{G})$ the convolution algebra of bounded complex regular measures on $\hat{G}$. We write the group operations in G and $\hat{G}$ additively, denote the identity element of G by 0, and write the action of a character $\gamma \in \hat{G}$ upon an element $x \in G$ as (x, γ). If $\mu \in M(\hat{G})$, we denote the *Fourier-Stieltjes transform* of μ by $\hat{\mu}(x) = \int_{\hat{G}} (x, \gamma)\, d\mu(\gamma)$. We denote Haar measure on G by dx and on $\hat{G}$ by $d\gamma$, and we define $B(G) = \{\hat{\mu} | \mu \in M(\hat{G})\}$, $P(G) = \{\hat{\mu} | \mu \in M(\hat{G})$ is a nonnegative measure$\}$, and $A_p(G) = \{\hat{\mu} | d\mu = F\, d\gamma$ for some $F \in L_1(\hat{G}) \cap L_p(\hat{G})\}$, $1 \le p \le \infty$. The *Fourier transform* of a function $f \in L_1(G)$ is denoted by $\hat{f}(\gamma) = \int_G (-x, \gamma) f(x)\, dx$. We refer to [7] for the basic facts about abstract harmonic analysis.

A complex valued function on G is said to be a *positive definite function* if

$$\sum_{i,j=1}^{n} \phi(x_i - x_j) c_i \bar{c}_j \ge 0 \tag{1.1}$$

for all $x_1, \ldots, x_n \in G$, all $c_1, \ldots, c_n \in \mathbf{C}$, and all $n = 1, 2, \ldots$. Bochner's classical characterization of $P(G)$ [2] is that it is precisely the set of continuous positive definite functions on G. Bochner's characterization of $B(G)$ [3] is of a very different nature, however. It does not use the elegant inequalities (1.1) and it does not identify the elements of $P(G) \cap B(G)$. Although (1.1) implies that ϕ is continuous on G if it is continuous at 0, the usual characterization of $B(G)$ requires the *a priori* assumption of continuity everywhere on G.

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It is the purpose of this note to present simple and similar characterizations of $A_p(G)$, $B(G)$, and $P(G)$ which are in the spirit of the inequalities (1.1). In the following section we list some basic facts about quadratic forms and state our characterization theorem; the proof and some remarks follow in the last three sections.

2. **Hermitian-symmetric inequalities.** If $A = (a_{ij})$ and $B = (b_{ij})$ are $n \times n$ complex matrices, we say that $A \succ_S B$ if B is symmetric and

$$\sum_{i,j=1}^{n} a_{ij} c_i \bar{c}_j \geq \left| \sum_{i,j=1}^{n} b_{ij} c_i c_j \right| \tag{2.1}$$

for all $c_1, \ldots, c_n \in \mathbf{C}$. If $C = (c_{ij})$ and $D = (d_{ij})$ are complex matrices with the same dimensions, we define the *Schur product* $C \circ D \equiv (c_{ij} d_{ij})$. We write $\det A$ for the determinant if A is a square matrix.

If $B \neq 0$, the relation $A \succ_S B$ imposes conditions on A which are stronger than ordinary positive definiteness. This relation is pregnant with interesting consequences, but we quote here only those results from [5] which are required for the purposes of this note.

Lemma 2.1. *Let A, B, C, D be $n \times n$ complex matrices.*

(a) *If $A = (a_{ij})$ and $B = (b_{ij})$, if B is symmetric, and if $A \succ_S B$, then*

(i) *A is a Hermitian positive semidefinite matrix,*

(ii) *$a_{ii} a_{jj} \geq \max(|a_{ij}|^2, |b_{ij}|^2)$ for all $i, j = 1, \ldots, n$,*

(iii) *$\det A \geq |\det B|$, and*

(iv) $$\left| \sum_{i,j=1}^{n} b_{ij} x_i y_j \right|^2 \leq \left(\sum_{i,j=1}^{n} a_{ij} x_i \bar{x}_j \right) \left(\sum_{i,j=1}^{n} a_{ij} y_i \bar{y}_j \right) \quad \textit{for all } x, y \in \mathbf{C}^n.$$

(b) *If $A \succ_S B$ and $C \succ_S D$, if B and D are symmetric, and if $\alpha_i, \beta_i \in \mathbf{C}$ are such that $\alpha_i \geq |\beta_i|$ for $i = 1, 2$ then*

(i) *$A \circ C \succ_S B \circ D$, and*

(ii) *$\alpha_1 A + \alpha_2 C \succ_S \beta_1 B + \beta_2 D$.*

We are here interested in a very special type of Hermitian-symmetric domination: If f and ϕ are complex valued functions on an Abelian group G, we say that $\phi(x - y) \succ_S f(x + y)$ on G if

$$\sum_{i,j=1}^{n} \phi(x_i - x_j) c_i \bar{c}_j \geq \left| \sum_{i,j=1}^{n} f(x_i + x_j) c_i c_j \right| \tag{2.2}$$

for all $x_1, \ldots, x_n \in G$, all $c_1, \ldots, c_n \in \mathbf{C}$, and all $n = 1, 2, \ldots$. A function ϕ is positive definite (1.1) if $\phi(x - y) \succ_S 0$ on G. As a direct corollary of Lemma 2.1 we have

Lemma 2.2. *Let G be an Abelian group, and let ϕ and f be complex valued functions on G.*

(a) *If* $\phi(x-y) \succ_S f(x+y)$ *on* G *then*

(i) ϕ *is a positive definite function on* G,

(ii) $\phi(0) \geq \max(|\phi(x)|, |f(x)|)$ *for all* $x \in G$, *and*

(iii) $|\Sigma_{i=1}^n f(x_i)c_i|^2 \leq \phi(0)\Sigma_{i,j=1}^n \phi(x_i - x_j)c_i\bar{c}_j$ *for all* $x_1, \ldots, x_n \in G$, *all* $c_1, \ldots, c_n \in \mathbf{C}$, *and all* $n = 1, 2, \ldots$.

(b) *If* $\phi_i(x-y) \succ_S f_i(x+y)$ *on* G *and if* $\alpha_i, \beta_i \in \mathbf{C}$ *are such that* $\alpha_i \geq |\beta_i|$ *for* $i = 1, 2$, *then*

(i) $(\phi_1\phi_2)(x-y) \succ_S (f_1f_2)(x+y)$ *on* G, *and*

(ii) $(\alpha_1\phi_1 + \alpha_2\phi_2)(x-y) \succ_S (\beta_1 f_1 + \beta_2 f_2)(x+y)$ *on* G.

(c) *If* G *is a topological Abelian group, if* $\phi(x-y) \succ_S f(x+y)$ *on* G, *and if* ϕ *and* f *are both continuous at* 0, *then* ϕ *and* f *are both uniformly continuous on* G.

(d) *If* G *is a locally compact Abelian group with Haar measure* dx, *if* $\phi(x-y) \succ_S f(x+y)$ *on* G, *and if* ϕ *and* f *are both continuous at* 0, *then*

(i) $$\left|\iint_{G\times G} f(x+y)h(x)h(y)\,dx\,dy\right| \leq \iint_{G\times G} \phi(x-y)h(x)\bar{h}(y)\,dx\,dy, \text{ and}$$

(ii) $$\left|\int_G f(x)h(x)\,dx\right|^2 \leq \phi(0)\iint_{G\times G} \phi(x-y)h(x)\bar{h}(y)$$

for all $h \in L_1(G)$.

Proof. If $\phi(x-y) \succ_S f(x+y)$ on G, then it is immediate from Lemma 2.1(a–i, ii) that ϕ is a positive definite function and that both ϕ and f are bounded by $\phi(0)$. Using Lemma 2.1(a–iv) we see that

$$(2.3)\quad \left|\sum_{i,j=0}^{n} f(x_i + x_j)c_i d_j\right|^2 \leq \left(\sum_{i,j=0}^{n} \phi(x_i - x_j)c_i\bar{c}_j\right)\left(\sum_{i,j=0}^{n} \phi(x_i - x_j)d_i\bar{d}_j\right)$$

for all $x_0, x_1, \ldots, x_n \in G$, all $c_0, d_0, \ldots, c_n, d_n \in \mathbf{C}$, and all $n = 0, 1, 2, \ldots$. The inequality in (a–iii) results if we set $x_0 = 0$, $d_0 = 1$, $c_0 = 0$, and $d_1 = d_2 = \cdots = d_n = 0$ in (2.3). The assertions in (b) follow directly from Lemma 2.1(b). If we choose $n = 3$, $x_1 = 0$, $x_2 = x$, and $x_3 = y$ in (2.2) and apply the determinant inequality in Lemma 2.1(a–iii) we obtain the inequality

$$\phi(0)[\phi^2(0) - |\phi(y)|^2] - 2\,\mathrm{Re}\,\phi(-x)\phi(x-y)[\phi(0) - \phi(y)] - \phi(0)|\phi(x) - \phi(x-y)|^2$$
$$\geq |-f(0)[f(x) - f(x+y)]^2 + [f(2y) - f(0)][f(0)f(2x) - f^2(x)]$$
$$+ [f(y) - f(0)][2f(x)f(x+y) - f(2x)\{f(y) + f(0)\}]|$$

which becomes

$$o(1) \geq \phi(0)|\phi(x) - \phi(x-y)|^2 + |f(0)||f(x) - f(x+y)|^2 \quad \text{as } y \to 0$$

if we invoke boundedness of f and ϕ and continuity at 0; the $o(1)$ terms are all uniform in x. If $\phi(0) = 0$ then $\phi(x) = 0 = f(x)$ for all $x \in G$ by (a–ii). If $f(0) = 0$, the above argument can be applied with ϕ and f replaced with

$\phi + 1$ and $f + 1$, respectively. Finally, (d) follows from (2.2) and (a–iii) with a limiting argument.

The above ideas permit us to state and prove our main result:

Theorem 2.3. *Let G be a locally compact Abelian group and let f be a complex valued function on G. Then*

(a) *$f \in P(G)$ if and only if f is continuous at 0 and $f(x-y) \succ_S f(x+y)$ on G,*

(b) *$f \in B(G)$ if and only if f is continuous at 0 and $\phi(x-y) \succ_S f(x+y)$ on G for some ϕ which is continuous at zero, and*

(c) *$f \in A_p(G)$ if and only if f is continuous at 0 and $\phi(x-y) \succ_S f(x+y)$ on G for some $\phi \in A_p(G)$, $1 \le p \le \infty$.*

If we denote by $P'(G)$, $B'(G)$, and $A_p'(G)$ the classes of functions which satisfy the conditions stated in (a), (b), and (c) above, respectively, then it follows from Lemma 2.2 that:

(i) Each function in $B'(G)$ is bounded and continuous;

(ii) $B'(G)$ and $A_p'(G)$ are linear spaces and $P'(G)$ is a convex cone;

(iii) $A_p'(G)$, $B'(G)$, and $P'(G)$ are all closed under ordinary multiplication of functions; and

(iv) $P'(G)$ is closed under uniform limits.

3. **Bochner's theorem and $P(G)$.** The fundamental characterization of $P(G)$ is due to Toeplitz [9] when $G = Z$, to Bochner [2] when $G = \mathbf{R}$, and to Weil [10] in the general case:

Theorem 3.1. *Let G be a locally compact Abelian group and let ϕ be a complex valued function on G. Then $\phi \in P(G)$ if and only if ϕ is a positive definite function on G which is continuous at 0.*

If $\phi = \hat{\mu}$ for some nonnegative measure $\mu \in M(\hat{G})$, then

$$\sum_{i,j=1}^{n} \phi(x_i - x_j)c_i\bar{c}_j = \sum_{i,j=1}^{n} \int_{\hat{G}} (x_i - x_j, y)c_i\bar{c}_j\, d\mu(y) = \int_{\hat{G}} \left|\sum_{i=1}^{n} c_i(x_i, y)\right|^2 d\mu(y)$$

$$\geq \left|\int_{\hat{G}} \left(\sum_{i=1}^{n} c_i(x_i, y)\right)^2 d\mu(y)\right| = \left|\int_{\hat{G}} \sum_{i,j=1}^{n} (x_i + x_j, y)c_ic_j\, d\mu(y)\right|$$

$$= \left|\sum_{i,j=1}^{n} \phi(x_i + x_j)c_ic_j\right|,$$

and hence $\phi(x-y) \succ_S \phi(x+y)$ on G.

Conversely, if $\phi(x-y) \succ_S \phi(x+y)$ on G then we know from Lemma 2.2(a–i) that ϕ is a positive definite function. This brief argument shows that:

If $\phi \in P(G)$ then $\phi(x-y) \succ_S \phi(x+y)$ on G. If ϕ is continuous at zero and $\phi(x-y) \succ_S 0$ on G, then $\phi \in P(G)$.

4. **The Bochner-Eberlein theorem and $B(G)$.** The fundamental characterization of $B(G)$ is due to Bochner [3] when $G = \mathbf{R}$ and to Eberlein [4] in the general case:

Theorem 4.1. *Let G be a locally compact Abelian group and let f be a complex valued function on G. Then $f \in B(G)$ if and only if f is continuous on G and*

$$\left|\sum_{i=1}^{n} c_i f(x_i)\right| \le K \sup_{\gamma\in\hat{G}} \left|\sum_{i=1}^{n} c_i (x_i, \gamma)\right| \tag{4.1}$$

for all $x_1, \ldots, x_n \in G$, all $c_1, \ldots, c_n \in \mathbf{C}$, and all $n = 1, 2, \ldots$.

If we denote the *total variation measure* of a measure $\mu \in M(\hat{G})$ by $|\mu|$ and its *total variation norm* by $\|\mu\| = \int_{\hat{G}} d|\mu|$, and if f satisfies (4.1), then it is known that $f = \hat{\mu}$ for some $\mu \in M(\hat{G})$ and $\|\mu\| = \inf\{K|$ equation (4.1) is satisfied$\}$.

If $f = \hat{\mu}$ for some $\mu \in M(\hat{G})$, then using the Jordan decomposition of the real and imaginary parts of μ we can write $f = a_1\mu_1 + \cdots + a_4\mu_4$ for some $a_1, \ldots, a_4 \in \mathbf{C}$ and some nonnegative measures $\mu_1, \ldots, \mu_4 \in M(\hat{G})$. Thus, for all $x_1, \ldots, x_n \in G$, all $c_1, \ldots, c_n \in \mathbf{C}$, and all $n = 1, 2, \ldots$ we have

$$\begin{aligned}
\left|\sum_{i,j=1}^{n} f(x_i + x_j)c_i c_j\right| &= \left|\sum_{k=1}^{4} a_k \sum_{i,j=1}^{n} \hat{\mu}_k(x_i + x_j)c_i c_j\right| \\
&\le \sum_{k=1}^{4} |a_k| \left|\sum_{i,j=1}^{n} \hat{\mu}_k(x_i + x_j)c_i c_j\right| \\
&\le \sum_{k=1}^{4} |a_k| \sum_{i,j=1}^{n} \hat{\mu}_k(x_i - x_j)c_i \bar{c}_j = \sum_{i,j=1}^{n} \phi(x_i - x_j)c_i \bar{c}_j
\end{aligned}$$

where $\phi \equiv |a_1|\hat{\mu}_1 + \cdots + |a_4|\hat{\mu}_4 \in P(G)$ and we have used Theorem 2.3(a).

Conversely, if f and ϕ are complex valued functions on G which are continuous at 0 and if $\phi(x-y) \succ_S f(x+y)$ on G, then by Lemma 2.2(a, c) and Theorem 3.1 we know that f and ϕ are uniformly continuous on G, $\phi \in P(G)$, and

$$\left|\sum_{i=1}^{n} f(x_i)c_i\right|^2 \le \phi(0) \sum_{i,j=1}^{n} \phi(x_i - x_j)c_i \bar{c}_j$$

for all $x_1, \ldots, x_n \in G$, all $c_1, \ldots, c_n \in \mathbf{C}$, and all $n = 1, 2, \ldots$. But since $\phi = \hat{\nu}$ for some nonnegative measure $\nu \in M(\hat{G})$ we also have

$$\sum_{i,j=1}^{n} \phi(x_i - x_j)c_i\bar{c}_j = \sum_{i,j=1}^{n} \int_{\hat{G}} (x_i - x_j, y)c_i\bar{c}_j \, d\nu(y)$$

$$= \int_{\hat{G}} \left|\sum_{i=1}^{n} c_i(x_i, y)\right|^2 d\nu(y) \leq \|\nu\| \sup_{y\in\hat{G}} \left|\sum_{i=1}^{n} c_i(x_i, y)\right|^2 .$$

Since $\phi(0) = \|\nu\|$ we conclude that (4.1) holds with $K = \phi(0)$. These observations show that:

> *If $f \in B(G)$ then there exists some $\phi \in P(G)$ such that $\phi(x - y) \succ_S f(x + y)$ on G. If f is continuous at zero and if there exists some ϕ continuous at zero such that $\phi(x - y) \succ_S f(x + y)$ on G, then $f \in B(G)$.*

Now suppose that $\phi(x - y) \succ_S f(x + y)$ on G and that both ϕ and f are continuous at zero; we know then that $\phi = \hat{\mu}$ and $f = \hat{\nu}$ for some $\mu, \nu \in M(\hat{G})$. If we write the inequality in Lemma 2.2(d–i) in terms of the Fourier transforms, we find that

$$\left|\int_{\hat{G}} (\hat{h}(y))^2 \, d\nu(y)\right| \leq \int_{\hat{G}} |\hat{h}(y)|^2 \, d\mu(y) \tag{4.2}$$

for all $h \in L_1(G)$. Let E be a compact subset of $\hat{G}$ and let V be any open set containing E. One knows [7, pp. 48–49] that there exists some $h \in L_1(G)$ such that $0 \leq \hat{h}(y) \leq 1$ for all $y \in \hat{G}$, $\hat{h} \equiv 1$ on E and $\hat{h} \equiv 0$ on the complement of V. Using such a function in inequality (4.2), we find that

$$\left|\nu(E) + \int_{V\sim E} (\hat{h}(y))^2 \, d\nu(y)\right| \leq \mu(E) + \int_{V\sim E} (\hat{h}(y))^2 \, d\mu(y)$$

where we denote the relative complement of E in V by $V \sim E$.

Thus for each compact set $E \subset \hat{G}$ and every open set $V \supset E$ we have $|\nu(E)| \leq \mu(E) + (\mu + |\nu|)(V \sim E)$. But since μ and ν are regular measures, the open set V can be chosen so as to make the last term arbitrarily small. We conclude that $|\nu(E)| \leq \mu(E)$ for every compact set E, and hence that $|\nu(B)| \leq \mu(B)$ and $|\nu|(B) \leq \mu(B)$ for every Borel set B. These measure inequalities imply that

$$\left|\int_{\hat{G}} g(y) \, d\nu(y)\right| \leq \int_{\hat{G}} |g(y)| \, d|\nu|(y) \leq \int_{\hat{G}} |g(y)| \, d\mu(y) \tag{4.3}$$

for all $g \in L_1(\hat{G}, |\nu|)$, and hence that

$$\left|\int_G f(x)h(x) \, dx\right| = \left|\int_{\hat{G}} \hat{H}(y) \, d\nu(y)\right| \leq \int_{\hat{G}} |\hat{H}(y)| \, d\mu(y) = \int_G \phi(x)h^{\dagger}(x) \, dx$$

for all $h \in L_1(G) \cap B(G)$. We write $H(x) \equiv h(-x)$ and denote the inverse

Fourier transform of the function $|\hat{H}|$ by $h^\dagger$; if $h \in L_1(G) \cap P(G)$ then $h^\dagger = h$.

The inequality $|\nu|(B) \leq \mu(B)$ implies that the measure ν is absolutely continuous with respect to μ; denote the Radon-Nikodym derivative by $d\nu/d\mu \equiv F$. Since $|\nu(B)| = |\int_B F(y)\, d\mu(y)| \leq \mu(B) = \int_B d\mu$ for every Borel set B, we must have $\mu\{y \in \hat{G} |\ |F(y)| > 1\} = 0$. Thus, we may assume that $F \in L_\infty(\hat{G})$ with $\|F\|_\infty \leq 1$. Conversely, if $d\nu = F\, d\mu$ for some bounded measurable function F with $|F| \leq 1$ on the support of μ, then it follows easily that $\hat{\mu}(x - y) \succ_S \hat{\nu}(x + y)$ on G. We summarize these observations in

Theorem 4.1. *Let ϕ and f be complex valued functions on a locally compact Abelian group G, and let both ϕ and f be continuous at zero. Then:*

(a) *$\phi(x - y) \succ_S f(x + y)$ on G if and only if there exists a nonnegative measure $\mu \in M(\hat{G})$ and a function $F \in L_\infty(\hat{G})$ such that $\mu\{|F| > 1\} = 0$, $\phi = \hat{\mu}$, and $f = \hat{\nu}$ where $d\nu = F\, d\mu$.*

(b) *$\phi(x - y) \succ_S f(x + y)$ on G if and only if $f = \hat{\nu}$ for some $\nu \in M(\hat{G})$ and $\phi = \hat{\mu}$ where $\mu = |\nu| + \lambda$ for some nonnegative measure $\lambda \in M(\hat{G})$.*

(c) *If $f = \hat{\nu}$ for some $\nu \in M(\hat{G})$, then $\|\nu\| = \inf\{\phi(0) | \phi(x - y) \succ_S f(x + y)$ on G and ϕ is continuous at zero$\}$.*

(d) *If $\phi(x - y) \succ_S f(x + y)$ on G, then $\phi = \hat{\mu}$ and $f = \hat{\nu}$ for some μ, $\nu \in M(\hat{G})$ with the support of ν contained in the support of μ.*

(e) *If $\phi(x - y) \succ_S f(x + y)$ on G then $|\int_G f(x) h(x)\, dx| \leq \int_G \phi(x) h^\dagger(x)\, dx$ for all $h \in L_1(G) \cap B(G)$.*

5. **The Berry-Ryan theorems and $A_p(G)$.** The fundamental characterization of $A_p(G)$ is due to Berry [1] when $G = \mathbf{R}$ and $p = 1$, and to Ryan [8] in the general case. Using Theorem 4.1, however, we can easily characterize $A_p(G)$, $1 \leq p \leq \infty$, using $A_p(G) \cap P(G)$. If $\phi(x - y) \succ_S f(x + y)$ on G and if both ϕ and f are continuous at zero, then we know $\phi = \hat{\mu}$ and $f = \hat{\nu}$ with $\mu, \nu \in M(\hat{G})$, μ nonnegative, and $d\nu = F\, d\mu$ for some $F \in L_\infty(\hat{G})$. If $\phi \in A_p(G)$ then μ must be absolutely continuous with respect to Haar measure and $d\mu = \Phi\, d\gamma$ for some $\Phi \in L_1(\hat{G}) \cap L_p(\hat{G})$. But then $d\nu = F\, d\mu = F\Phi\, d\gamma$ and $F\Phi \in L_1(\hat{G}) \cap L_p(\hat{G})$ so $f \in A_p(G)$.

Conversely, if $f \in A_p(G)$ then $f = \hat{\nu}$ where $d\nu = \Phi\, d\gamma$ and $\Phi \in L_1(\hat{G}) \cap L_p(\hat{G})$. But if we set $\phi \equiv \hat{\mu}$ where $d\mu \equiv d|\nu| = |\Phi|\, d\gamma$, then $\phi \in A_p(G)$ and $\phi(x - y) \succ_S f(x + y)$ by Theorem 4.1(b). These observations show that:

> *If $f \in A_p(G)$ then there exists some $\phi \in A_p(G) \cap P(G)$ such that $\phi(x - y) \succ_S f(x + y)$ on G. If f is continuous at zero and if there exists some $\phi \in A_p(G)$ such that $\phi(x - y) \succ_S f(x + y)$ on G, then $f \in A_p(G)$.*

It should be noted that it may often be easier to tell whether a given function $\phi \in P(G)$ is also in $A_p(G)$ than to tell whether a function $f \in B(G)$ is also in $A_p(G)$. For example, when $G = \mathbf{R}$ and $p = 1$ Pólya's criterion [6] is a simple sufficient condition: ϕ is real valued, even, continuous, convex on $(0, \infty)$, and $\phi(x) \to 0$ as $x \to \infty$.

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