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RELATIVELY UNIFORM BANACH LATTICES[1]

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ABSTRACT. Sequential relative uniform and norm convergence agree in a Banach lattice, if and only if it is equivalent to an M space.

Let $(E, \leq, \|\cdot\|)$ be a Banach lattice. The sequence $\{f_n\}$ is said to be *relatively uniformly* (abbreviated ru) *convergent* to f, if there exists $u > 0$, $\lambda_n \in \mathbf{R}$, $\lambda_n \downarrow 0$, such that $|f_n - f| \leq \lambda_n u$ for all n. If there exist an increasing sequence $\{g_n\}$ and a decreasing sequence $\{h_n\}$ such that $g_n \leq f_n \leq h_n$ for all n, and $\bigvee g_n = \bigwedge h_n = f$, we say $\{f_n\}$ is *order* (abbreviated o) *convergent* to f. We say two Banach lattices are *equivalent* if there exists a vector lattice isomorphism under which the norms are equivalent. Call E *relatively uniform* if norm and ru convergence agree, *for sequences*, on E.

Firstly we indicate why the corresponding notion for nets is rather trivial. We say the net $\{f_\alpha\}_{\alpha \in A}$ ru converges to f if there exists $u > 0$, such that given any $\epsilon > 0$ there exists $\alpha_0 \in A$ with $|f_\alpha - f| \leq \epsilon u$ for all $\alpha \geq \alpha_0$.

Proposition 1. *E is net relatively uniform if and only if E contains a strong unit.*

Proof. Suppose E is net relatively uniform. If $\alpha, \beta \in E \backslash \{0\}$ write $\alpha \preccurlyeq \beta$ to mean $\|\alpha\| \geq \|\beta\|$. Let $f_\alpha = \alpha$, then $\|\cdot\|$-lim $f_\alpha = 0$, so ru-lim $f_\alpha = 0$. Hence E contains a strong unit.

Conversely suppose E contains a strong unit e. The norms $\|\cdot\|$ and $\|\cdot\|_e$ are equivalent, by Birkhoff [1, Lemma 5, p. 367], where $\|f\|_e = \inf\{\lambda: |f| \leq \lambda e\}$. □

From now on we only consider sequential convergence.

The following result is contained, implicitly, in Leader [3].

Lemma 1 (Leader). *If E is relatively uniform then it is equivalent to an M space.*

Proof. By [3, Theorem 4] it follows that $\|\cdot\|$ is equivalent to an M norm if and only if, for every sequence $\{p(n)\}$ of positive integers, and for every sequence $\{f_n\}$ such that $\|f_n\| \to 0$, we have

(1) $$\| |f_n| \vee |f_{n+1}| \vee \cdots \vee |f_{n+p(n)}| \| \to 0.$$

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If E is relatively uniform and $\|f_n\| \to 0$, then $|f_n| \vee |f_{n+1}| \vee \cdots \vee |f_{n+p(n)}| \leq \lambda_n u$, hence (1) holds. □

Lemma 2. *In an M space a norm convergent sequence has a supremum.*

Proof. Let $\{f_n\}$ be a norm convergent sequence. Denote $f_1 \vee \cdots \vee f_n$ by g_n. We show $\{g_n\}$ is Cauchy. Let $n > m$, then

$$\begin{aligned} g_n - g_m &= g_m \vee f_{m+1} \vee \cdots \vee f_n - g_m \\ &= (f_{m+1} - g_m)^+ \vee \cdots \vee (f_n - g_m)^+. \end{aligned}$$

Now

$$f_i - g_m = f_i - f_1 \vee \cdots \vee f_m = (f_i - f_1) \wedge \cdots \wedge (f_i - f_m),$$

so $(f_i - g_m)^+ \leq (f_i - f_m)^+$. Hence

$$\begin{aligned} \|g_n - g_m\| &= \left\| \bigvee_{i=m+1}^{n} (f_i - g_m)^+ \right\| \leq \left\| \bigvee_{i=m+1}^{n} (f_i - f_m)^+ \right\| \\ &= \bigvee_{i=m+1}^{n} \|(f_i - f_m)^+\| \leq \bigvee_{i=m+1}^{n} \|f_i - f_m\|. \end{aligned}$$

Thus $\{g_n\}$ is Cauchy. Let $\|\cdot\|$-lim $g_n = g$; it then follows, from the fact that the positive cone is (norm) closed, that $g = \bigvee_{n=1}^{\infty} f_n$. □

Proposition 2. *The following statements are pairwise equivalent:*

(i) *E is relatively uniform;*

(ii) *E is equivalent to an M space;*

(iii) *$\{f_n\}$ norm converges implies that $\bigvee f_n$ exists;*

(iv) *$\{f_n\}$ norm converges implies that $\bigwedge f_n$ exists;*

(v) *$\{f_n\}$ norm converges implies that $\{f_n\}$ order converges.*

Proof. (i) implies (ii) is Lemma 1, (ii) implies (iii) is Lemma 2.

(iii) implies (v). If $\|f_n\| \to 0$ then $u = \bigvee_{\|f_n\|>0} \|f_n\|^{-1/2} |f_n|$ exists. So f_n ru converges to 0. Since E is archimedean, (v) follows.

(iii) and (iv) are equivalent. If $\{f_n\}$ norm converges then so does $\{-f_n\}$, and $\bigvee -f_n = -\bigwedge f_n$.

(v) implies (i). Suppose $\|f_n\| \to 0$ then $\|f_n\|^{-1/2} |f_n|$ norm converges to 0, so by (v) there exists $v \geq \|f_n\|^{-1/2} |f_n|$, hence (i) holds (we may assume without loss of generality that $\|f_n\| > 0$ for all n). □

A Banach lattice is called an L^p *space* if $|f| \wedge |g| = 0$ implies that $\|f + g\| = (\|f\|^p + \|g\|^p)^{1/p}$ where $1 \leq p < \infty$ (Birkhoff [1, p. 378]).

Corollary 1. *An infinite dimensional L^p space is not relatively uniform, and is not equivalent to an M space.*

Proof. Suppose there exists an infinite dimensional relatively uniform L^p space, E. By Zaanen [4, Theorem 4.3] there exists a sequence $\{f_n\}$ of strictly positive pairwise orthogonal elements in E. We may assume that $\|f_n\| = n^{-1/p}$. So ru-lim $f_n = 0$, and hence there exists $u \in E$ such that $f_n \leq u$ for all n. So

$$\|u\| \geq \|f_1 \vee \cdots \vee f_n\| = \|f_1 + \cdots + f_n\|$$

$$= (\|f_1\|^p + \cdots + \|f_n\|^p)^{1/p} = \left(\sum_{k=1}^{n} \frac{1}{k} \right)^{1/p} \longrightarrow \infty,$$

giving the required contradiction. The rest follows from Proposition 2. □

Corollary 2. *In an infinite dimensional L^p space every l-ideal, generated by countably many elements, is proper.*

Proof. Suppose E is an infinite dimensional L^p space, containing a sequence $\{f_n > 0\}$ which is contained in no proper l-ideal. Let $e = \sum_{n=1}^{\infty} n^{-2} \|f_n\|^{-1} f_n$, then e is a strong unit. With the equivalent norm $\|\cdot\|_e$ (see Proposition 1) E becomes an M space, with unity e. The rest follows by Corollary 1. □

In the case of l^1 the above reduces to the well-known result, that even with countably many 'test' series the comparison test is not effective on all positive term series; see Knopp [2].

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