A CARLESON MEASURE THEOREM FOR BERGMAN SPACES

WILLIAM W. HASTINGS

ABSTRACT. Let \( \mu \) be a finite, positive measure on \( U^n \), the unit polydisc in \( \mathbb{C}^n \), and let \( \sigma_n \) be the \( 2n \)-dimensional Lebesgue volume measure on \( U^n \). For \( 1 \leq p \leq q < \infty \), a necessary and sufficient condition on \( \mu \) is given in order that
\[
\left\{ \int_{U^n} f^q(z) \, d\mu(z) \right\}^{1/q} \leq C \left\{ \int_{U^n} f^p(z) \, d\sigma_n(z) \right\}^{1/p}
\]
for every positive \( n \)-subharmonic function \( f \) on \( U^n \).

A theorem of Carleson [1], [2] as generalized by Duren [3] characterizes those positive measures \( \mu \) on \( |z| < 1 \) for which the \( H^p \) norm dominates the \( L^q \) norm of elements of \( H^p \). The purpose of this note is to prove an analogous result with \( H^p \) replaced by \( A^p \), the Bergman space of functions \( f \) which are analytic in \( |z| < 1 \) and for which \( \int_0^1 \int_0^{2\pi} |f(re^{i\theta})|^p \, r \, d\theta \, dr < \infty \). Actually, the result is more general in that it applies to positive \( n \)-subharmonic functions and positive measures on the unit polydisc in \( \mathbb{C}^n \). I wish to express my gratitude to Professor Allen Shields for suggesting this problem and guiding me to its solution. My thanks also go to the referee for outlining a correction to an error in my proof.

First, some notation and a definition. Let
\[
U^n = \{ z = (z_1, \ldots, z_n) \in \mathbb{C}^n : |z_j| < 1, 1 \leq j \leq n \}
\]
and let \( \sigma_n \) be the \( 2n \)-dimensional Lebesgue volume measure restricted to \( U^n \), normalized so that \( U^n \) has measure one. Suppose that \( f \) is upper semicontinuous on \( U^n \). Then we say that \( f \) is \( n \)-subharmonic provided that \( f \) is subharmonic in each variable separately (cf. [5, p. 39]).

**Theorem.** Let \( \mu \) be a finite, positive measure on \( U^n \), and suppose \( 1 \leq p \leq q < \infty \). Then there exists a constant \( C > 0 \) such that
\[
\left\{ \int_{U^n} f^q(z) \, d\mu(z) \right\}^{1/q} \leq C \left\{ \int_{U^n} f^p(z) \, d\sigma_n(z) \right\}^{1/p}
\]
for every positive \( n \)-subharmonic function \( f \) on \( U^n \) if and only if there exists a constant \( C' > 0 \) such that
\[
\mu(S) \leq C' \left( \prod_{j=1}^n \delta_j \right)^{2q/p}
\]
for every set \( S \) of the form
\[
S = \{ z = (r_1 e^{i\theta_1}, \ldots, r_n e^{i\theta_n}) : 1 - \delta_j \leq r_j < 1, \theta_j^0 \leq \theta_j \leq \theta_j^0 + \delta_j, 1 \leq j \leq n \}.
\]
Proof. Suppose that $0 < p < q < \infty$, and suppose that (1) holds for every positive $n$-subharmonic function $f$ on $U^n$. Let $S$ be a set of the form (3). Let

$$
\alpha_j = (1 - \delta_j) \exp\{i(\theta_j + \delta_j/2)\}, \quad 1 \leq j \leq n,
$$

and set

$$
f(z) = \prod_{j=1}^n |1 - \alpha_j z_j|^{-4/p}.
$$

A geometric argument [4, p. 157] shows that in $S$,

$$
f_p(z) \geq c_1 \prod_{j=1}^n \delta_j^{-4}.
$$

Therefore,

$$
c_1^{q/p} \prod_{j=1}^n \delta_j^{-4q/p} \mu(S) \leq \int f^q \, d\mu \leq C q \left\{ \int f^p \, d\sigma \right\}^{q/p} \leq C q \prod_{j=1}^n \delta_j^{-2q/p}
$$

and (2) holds with $C' = c_1^{-q/p} C q$.

Conversely, suppose that (2) holds for every set $S$ of the form (3). For $m = (m_1, \ldots, m_n) \in \mathbb{Z}^n$ and $k = (k_1, \ldots, k_n) \in \mathbb{Z}^n$ with $m_j \geq 0$ and $1 \leq k_j \leq 2^{m_j + 4}$, $1 \leq j \leq n$, set

$$
T_{mk} = \{ z = (r_1 e^{i\theta_1}, \ldots, r_n e^{i\theta_n}) : 1 - 2^{-m_j} \leq r_j < 1 - 2^{-m_j - 1}, \quad 2k_j \pi / 2^{m_j + 4} \leq \theta_j < 2(k_j + 1)\pi / 2^{m_j + 4}, \quad 1 \leq j \leq n \},
$$

and set $z^m = (z_1^m, \ldots, z_n^m)$, where

$$
z_j^m = (1 - 2^{-m_j}) \exp\{2(k_j + 1/4)\pi i / 2^{m_j + 4}\}, \quad 1 \leq j \leq n.
$$

Note that

$$
\mu(T_{mk}) \leq C \left( \prod_{j=1}^n 2^{-m_j} \right)^{2q/p} \quad \text{and} \quad \max_{z \in T_{mk}} |z_j - z_j^m| < \frac{11 - 2^{-m_j}}{16}, \quad 1 \leq j \leq n.
$$

Now suppose that $f$ is positive and $n$-subharmonic in $U^n$. For $(6/8)2^{-m_j} \leq \rho_j \leq (7/8)2^{-m_j}$ and $z \in T_{mk}$, repeated application of Harnack’s inequality yields

$$
f_p(z) \leq (2\pi)^{-n} \prod_{j=1}^n \left( \frac{\rho_j + |z_j - z_j^m|}{\rho_j - |z_j - z_j^m|} \right) \cdot \int_{0}^{2\pi} \cdots \int_{0}^{2\pi} f_p(z_1^m + \rho_1 e^{i\theta_1}, \ldots, z_n^m + \rho_n e^{i\theta_n}) \, d\theta_1 \cdots d\theta_n
$$

$$
\leq c_1 (2\pi)^{-n} \int_{0}^{2\pi} \cdots \int_{0}^{2\pi} f_p(z_1^m + \rho_1 e^{i\theta_1}, \ldots, z_n^m + \rho_n e^{i\theta_n}) \, d\theta_1 \cdots d\theta_n.
$$

Hence, for $z \in T_{mk}$,
\[ f^p(z) = c_1 \left( \prod_{j=1}^{n} 4^{m_j} \right) \int (7/8)^{-m_1} \cdots \int (7/8)^{-m_n} f^p(z) \rho_1 \cdots \rho_n \varphi_1 \cdots \varphi_n \]

\[ \leq c_1 c_2 \left( \prod_{j=1}^{n} 4^{m_j} \right) \int_{U_{m_k}} f^p \, d\sigma \]

where

\[ U_{m_k} = \{ z = (z_1, \ldots, z_n) \in \mathbb{C}^n : |z_j - z_{m_k}^j| \leq (7/8)^{-m_j}, \ 1 \leq j \leq n \}. \]

Since \( z \) was arbitrary in \( T_{m_k} \), we have

\[ \int_{U_n} f^q \, d\mu = \sum_{m=(m_1, \ldots, m_n)} c_{m k} \sum_{k=(k_1, \ldots, k_n)} \int_{T_{m_k}} f^q \, d\mu \]

\[ \leq \sum_{m} \sum_{k} \mu(T_{m_k}) \left\{ c_1 c_2 \left( \prod_{i=1}^{n} 4^{m_i} \right) \int_{U_{m_k}} f^p \, d\sigma \right\}^{q/p} \]

\[ \leq c_3 \sum_{m} \sum_{k} \left\{ \int_{U_{m_k}} f^p \, d\sigma \right\}^{q/p} \]

Fix \( m^0 = (m_1^0, \ldots, m_n^0) \) and \( k^0 = (k_1^0, \ldots, k_n^0) \) with \( m_j^0 \geq 0 \) and \( 1 \leq k_j^0 \leq 2m_j^0+4 \), \( 1 \leq j \leq n \). We claim that \( T_{m^0 k^0} \) intersects \( U_{m_k} \) for at most \( N = (5.57)^n \) choices of the pair \((m, k)\). Assume first that \( m_j^0 \geq 1 \). If

\[ z = (r_1 e^{i\theta_1}, \ldots, r_n e^{i\theta_n}) \in T_{m^0 k^0}, \]

then \( 1-2^{-m_j^0} \leq r_j < 1-2^{-m_j^0-1} \), while if \( z \in U_{m_k} \), then \( 1-2^{-m_j+1} < r_j \leq 1-2^{-m_j-3} \). Hence, if \( z \in T_{m^0 k^0} \cap U_{m_k} \), then combination of the two inequalities shows that \( m_j^0 - 3 \leq m_j \leq m_j^0 + 2 \). Then \( m_j \) can be one of at most five different values. Similarly, if \( z \in T_{m^0 k^0} \), then we may assume that

\[ 2k_j^0 \pi/2m_j^0+4 \leq \theta_j < 2(k_j^0 + 1)\pi/2m_j^0+4. \]

A little geometry shows that if \( z \in U_{m_k} \), then

\[ \frac{2(k_j^0 + \frac{1}{2})\pi}{2m_j^0+4} - \frac{7\pi}{8} 2^{-m_j} \leq \theta_j + 2ln(2) \leq \frac{2(k_j^0 + \frac{1}{2})\pi}{2m_j^0+4} + \frac{7\pi}{8} 2^{-m_j} \]

where \( l \) is either 0, 1 or -1. Combination of these two inequalities yields

\[ \frac{0}{2} \leq k_j + \frac{1}{2} \leq 2 \]

\[ 1 \leq l \leq 7. \]
Hence, if \( z \in T_{m_0 k_0} \cap U_{m k} \), then \( k_j \) can be one of at most \( 3(15 + 2m_j - m_j^0) \leq 57 \) different values. (The factor of 3 reflects the three possible values of \( l \).) This establishes the claim for \( m_j^0 \geq 1 \), but if \( m_j^0 = 0 \) for any \( j \) in the above counting procedure, then we would have even fewer intersections. Therefore,

\[
\sum_{m_j \geq 0} \sum_{1 \leq k_j \leq m_j^0 + 4} \left\{ \int_{U_{m k}} f^p \, d\sigma_n \right\}^{q/p} \leq \left\{ \sum_{m, k} \int_{U_{m k}} f^p \, d\sigma_n \right\}^{q/p}
\]

and the proof is complete.

Remark. The Theorem also holds for \( 0 < p \leq q < \infty \) if we require that \( f = |g|^2 \), where \( g \) is holomorphic in \( U^n \). In this case \( f^p \) is \( n \)-subharmonic, and so the proof is the same.

As in [3] two inequalities follow immediately.

Corollary. For a positive subharmonic function \( f \) on \( |z| < 1 \) and for \( 1 \leq p \leq q < \infty \),

\[
\left\{ \int_0^1 f^q(r)(1-r)^{2(q/p)-1} \, dr \right\}^{1/q} \leq C \left\{ \int_{|z| < 1} f^p(z) \, d\sigma_1(z) \right\}^{1/p}
\]

and

\[
\left\{ \int_0^1 (1-r)^{2(q/p)-2} \int_0^{2\pi} f^{q(\theta)} \, d\theta d\theta \right\}^{1/q} \leq C \left\{ \int_{|z| < 1} f^p(z) \, d\sigma_1(z) \right\}^{1/p}
\]

where the constants \( C \) and \( C' \) may be chosen independently of \( f \).

For another application, suppose \( |z_j|_{j=1}^\infty \) is a sequence of distinct points in \( |z| < 1 \). Let \( \mu \) be the point measure defined by \( \mu(z_j) = (1 - |z_j|^2)^2 \), \( j \geq 1 \), and \( \mu(U \setminus \{z_j\}^\infty_{j=1}) = 0 \). For \( f \in A^p (p > 0) \) let \( T_p f \) be the sequence \( \{(z_j)(1 - |z_j|^2)^{2/p} \}^\infty_{j=1} \).

Corollary. For \( 0 < p < \infty \), \( T(A^p) \subseteq L^p \) if and only if \( \mu(S) \leq c\delta_1^2 \) for every set \( S \) of the form (3) with \( n = 1 \).
REFERENCES


DEPARTMENT OF MATHEMATICS, UNIVERSITY OF MICHIGAN, ANN ARBOR, MICHIGAN 48104

Current address: Department of Mathematics, Fordham University, Bronx, New York 10458