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A CARLESON MEASURE THEOREM FOR BERGMAN SPACES

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ABSTRACT. Let μ be a finite, positive measure on U^n, the unit polydisc in $\mathbf{C}^n$, and let σ_n be $2n$-dimensional Lebesgue volume measure on U^n. For $1 \le p \le q < \infty$ a necessary and sufficient condition on μ is given in order that $\{\int_{U^n} f^q(z)\,d\mu(z)\}^{1/q} \le C\{\int_{U^n} f^p(z)\,d\sigma_n(z)\}^{1/p}$ for every positive n-subharmonic function f on U^n.

A theorem of Carleson [1], [2] as generalized by Duren [3] characterizes those positive measures μ on $|z| < 1$ for which the H^p norm dominates the $L^q(\mu)$ norm of elements of H^p. The purpose of this note is to prove an analogous result with H^p replaced by A^p, the Bergman space of functions f which are analytic in $|z| < 1$ and for which $\int_0^1 \int_0^{2\pi} |f(re^{i\theta})|^p r\,d\theta\,dr < \infty$. Actually, the result is more general in that it applies to positive n-subharmonic functions and positive measures on the unit polydisc in $\mathbf{C}^n$. I wish to express my gratitude to Professor Allen Shields for suggesting this problem and guiding me to its solution. My thanks also go to the referee for outlining a correction to an error in my proof.

First, some notation and a definition. Let

$$U^n = \{z = (z_1, \dots, z_n) \in \mathbf{C}^n : |z_j| < 1,\ 1 \le j \le n\},$$

and let σ_n be $2n$-dimensional Lebesgue volume measure restricted to U^n, normalized so that U^n has measure one. Suppose that f is upper semicontinuous on U^n. Then we say that f is n-subharmonic provided that f is subharmonic in each variable separately (cf. [5, p. 39]).

Theorem. *Let μ be a finite, positive measure on U^n, and suppose $1 \le p \le q < \infty$. Then there exists a constant $C > 0$ such that*

$$\left\{\int_{U^n} f^q(z)\,d\mu(z)\right\}^{1/q} \le C\left\{\int_{U^n} f^p(z)\,d\sigma_n(z)\right\}^{1/p} \tag{1}$$

for every positive n-subharmonic function f on U^n if and only if there exists a constant $C' > 0$ such that

$$\mu(S) \le C'\left(\prod_{j=1}^n \delta_j\right)^{2q/p} \tag{2}$$

for every set S of the form

$$S = \{z = (r_1e^{i\theta_1}, \dots, r_ne^{i\theta_n}) : 1 - \delta_j \le r_j < 1,\ \theta_j^0 \le \theta_j \le \theta_j^0 + \delta_j,\ 1 \le j \le n\}. \tag{3}$$

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Proof. Suppose that $0 < p \leq q < \infty$, and suppose that (1) holds for every positive n-subharmonic function f on U^n. Let S be a set of the form (3). Let

$$\alpha_j = (1 - \delta_j) \exp\{i(\theta_j^0 + \delta_j/2)\}, \qquad 1 \leq j \leq n,$$

and set

$$f(z) = \prod_{j=1}^{n} |1 - \overline{\alpha}_j z_j|^{-4/p}.$$

A geometric argument [4, p. 157] shows that in S,

$$f^p(z) \geq c_1 \prod_{j=1}^{n} \delta_j^{-4}.$$

Therefore,

$$c_1^{q/p} \prod_{j=1}^{n} \delta_j^{-4q/p} \mu(S) \leq \int f^q \, d\mu \leq C^q \left\{ \int f^p \, d\sigma_n \right\}^{q/p} \leq C^q \prod_{j=1}^{n} \delta_j^{-2q/p}$$

and (2) holds with $C' = c_1^{-q/p} C^q$.

Conversely, suppose that (2) holds for every set S of the form (3). For $m = (m_1, \ldots, m_n) \in \mathbf{Z}^n$ and $k = (k_1, \ldots, k_n) \in \mathbf{Z}^n$ with $m_j \geq 0$ and $1 \leq k_j \leq 2^{m_j+4}$, $1 \leq j \leq n$, set

$$T_{mk} = \{z = (r_1 e^{i\theta_1}, \ldots, r_n e^{i\theta_n}) : 1 - 2^{-m_j} \leq r_j < 1 - 2^{-m_j-1},$$

$$2k_j \pi / 2^{m_j+4} \leq \theta_j < 2(k_j + 1)\pi / 2^{m_j+4}, \; 1 \leq j \leq n\},$$

and set $z^{mk} = (z_1^{mk}, \ldots, z_n^{mk})$, where

$$z_j^{mk} = (1 - 2^{-m_j}) \exp\{2(k_j + \tfrac{1}{2})\pi i / 2^{m_j+4}\}, \qquad 1 \leq j \leq n.$$

Note that

$$\mu(T_{mk}) \leq C \left(\prod_{j=1}^{n} 2^{-m_j} \right)^{2q/p} \quad \text{and} \quad \max_{z \in T_{mk}} |z_j - z_j^{mk}| < \frac{11}{16} 2^{-m_j}, \qquad 1 \leq j \leq n.$$

Now suppose that f is positive and n-subharmonic in U^n. For $(6/8)2^{-m_j} \leq \rho_j \leq (7/8)2^{-m_j}$ and $z \in T_{mk}$, repeated application of Harnack's inequality yields

$$f^p(z) \leq (2\pi)^{-n} \prod_{j=1}^{n} \left(\frac{\rho_j + |z_j - z_j^{mk}|}{\rho_j - |z_j - z_j^{mk}|} \right)$$

$$\cdot \int_0^{2\pi} \cdots \int_0^{2\pi} f^p(z_1^{mk} + \rho_1 e^{i\theta_1}, \ldots, z_n^{mk} + \rho_n e^{i\theta_n}) \, d\theta_1 \cdots d\theta_n$$

$$\leq c_1 (2\pi)^{-n} \int_0^{2\pi} \cdots \int_0^{2\pi} f^p(z_1^{mk} + \rho_1 e^{i\theta_1}, \ldots, z_n^{mk} + \rho_n e^{i\theta_n}) \, d\theta_1 \cdots d\theta_n.$$

Hence, for $z \in T_{mk}$,

$$f^p(z) = c_2\left(\prod_{j=1}^{n} 4^{m_j}\right)\int_{(6/8)2^{-m_1}}^{(7/8)2^{-m_1}} \cdots \int_{(6/8)2^{-m_n}}^{(7/8)2^{-m_n}} f^p(z)\rho_1\cdots\rho_n d\rho_1\cdots d\rho_n$$

$$\leq c_1c_2\left(\prod_{j=1}^{n} 4^{m_j}\right)\int_{U_{mk}} f^p\, d\sigma_n$$

where

$$U_{mk} = \{z = (z_1, \ldots, z_n) \in \mathbf{C}^n : |z_j - z_j^{mk}| \leq (7/8)2^{-m_j},\ 1 \leq j \leq n\}.$$

Since z was arbitrary in T_{mk}, we have

$$\int_{U^n} f^q\, d\mu = \sum_{\substack{m=(m_1,\ldots,m_n)\\ m_j\geq 0}} \sum_{\substack{k=(k_1,\ldots,k_n)\\ 1\leq k_j\leq 2^{m_j+4}}} \int_{T_{mk}} f^q\, d\mu$$

$$\leq \sum_m \sum_k \mu(T_{mk})\left\{c_1c_2\left(\prod_{i=1}^{n} 4^{m_i}\right)\int_{U_{mk}} f^p\, d\sigma_n\right\}^{q/p}$$

$$\leq c_3 \sum_m \sum_k \left\{\int_{U_{mk}} f^p\, d\sigma_n\right\}^{q/p}.$$

Fix $m^0 = (m_1^0, \ldots, m_n^0)$ and $k^0 = (k_1^0, \ldots, k_n^0)$ with $m_j^0 \geq 0$ and $1 \leq k_j^0 \leq 2^{m_j^0+4}$, $1 \leq j \leq n$. We claim that $T_{m^0k^0}$ intersects U_{mk} for at most $N = (5\cdot 57)^n$ choices of the pair (m, k). Assume first that $m_j^0 \geq 1$. If

$$z = (r_1e^{i\theta_1}, \ldots, r_ne^{i\theta_n}) \in T_{m^0k^0},$$

then $1 - 2^{-m_j^0} \leq r_j < 1 - 2^{-m_j^0-1}$, while if $z \in U_{mk}$, then $1 - 2^{-m_j+1} < r_j \leq 1 - 2^{-m_j-3}$. Hence, if $z \in T_{m^0k^0} \cap U_{mk}$, then combination of the two inequalities shows that $m_j^0 - 3 \leq m_j < m_j^0 + 2$. Then m_j can be one of at most five different values. Similarly, if $z \in T_{m^0k^0}$, then we may assume that

$$2k_j^0\pi/2^{m_j^0+4} \leq \theta_j < 2(k_j^0 + 1)\pi/2^{m_j^0+4}.$$

A little geometry shows that if $z \in U_{mk}$, then

$$\frac{2(k_j + \frac{1}{2})\pi}{2^{m_j+4}} - \frac{7\pi}{8}2^{-m_j} \leq \theta_j + 2l\pi \leq \frac{2(k_j + \frac{1}{2})\pi}{2^{m_j+4}} + \frac{7\pi}{8}2^{-m_j}$$

where l is either 0, 1 or -1. Combination of these two inequalities yields

$$2^{m_j-m_j^0}k_j^0 + 2^{m_j+4}l - 7 \leq k_j + \tfrac{1}{2} \leq 2^{m_j-m_j^0}k_j^0 + 2^{m_j-m_j^0} + 2^{m_j+4}l + 7.$$

Hence, if $z \in T_{m^0 k^0} \cap U_{mk}$, then k_j can be one of at most $3(15 + 2^{m_j - m_j^0})$ ≤ 57 different values. (The factor of 3 reflects the three possible values of l.) This establishes the claim for $m_j^0 \geq 1$, but if $m_j^0 = 0$ for any j in the above counting procedure, then we would have even fewer intersections. Therefore,

$$\sum_{\substack{m=(m_1,\dots,m_n)\\ m_j \geq 0}} \sum_{\substack{k=(k_1,\dots,k_n)\\ 1 \leq k_j \leq 2^{m_j+4}}} \left\{ \int_{U_{mk}} f^p \, d\sigma_n \right\}^{q/p} \leq \left\{ \sum_{m,k} \int_{U_{mk}} f^p \, d\sigma_n \right\}^{q/p}$$

$$= \left\{ \sum_{m,k} \sum_{m^0, k^0} \int_{T_{m^0 k^0} \cap U_{mk}} f^p \, d\sigma_n \right\}^{q/p}$$

$$= \left\{ \sum_{m^0, k^0} \sum_{m,k} \int_{T_{m^0 k^0} \cap U_{mk}} f^p \, d\sigma_n \right\}^{q/p}$$

$$\leq \left\{ N \sum_{m^0, k^0} \int_{T_{m^0 k^0}} f^p \, d\sigma_n \right\}^{q/p} = N^{q/p} \left\{ \int_{U^n} f^p \, d\sigma_n \right\}^{q/p},$$

and the proof is complete.

Remark. The Theorem also holds for $0 < p \leq q < \infty$ if we require that $f = |g|$, where g is holomorphic in U^n. In this case f^p is n-subharmonic, and so the proof is the same.

As in [3] two inequalities follow immediately.

Corollary. *For a positive subharmonic function f on $|z| < 1$ and for $1 \leq p \leq q < \infty$,*

$$\left\{ \int_0^1 f^q(r)(1-r)^{2(q/p)-1} \, dr \right\}^{1/q} \leq C \left\{ \int_{|z|<1} f^p(z) \, d\sigma_1(z) \right\}^{1/p}$$

and

$$\left\{ \int_0^1 (1-r)^{2(q/p)-2} \int_0^{2\pi} f^q(re^{i\theta}) \, d\theta \, dr \right\}^{1/q} \leq C' \left\{ \int_{|z|<1} f^p(z) \, d\sigma_1(z) \right\}^{1/p}$$

where the constants C and C' may be chosen independently of f.

For another application, suppose $\{z_j\}_{j=1}^{\infty}$ is a sequence of distinct points in $|z| < 1$. Let μ be the point measure defined by $\mu\{z_j\} = (1 - |z_j|^2)^2$, $j \geq 1$, and $\mu(U \setminus \{z_j\}_{j=1}^{\infty}) = 0$. For $f \in A^p$ $(p > 0)$ let $T_p f$ be the sequence $\{f(z_j)(1 - |z_j|^2)^{2/p}\}_{j=1}^{\infty}$.

Corollary. *For $0 < p < \infty$, $T_p(A^p) \subset l^p$ if and only if $\mu(S) \leq c\delta_1^2$ for every set S of the form* (3) *with $n = 1$.*

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