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GENERALIZED ALGEBRAIC OPERATORS[1]

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ABSTRACT. A class of polynomially bounded operators satisfying an H^∞ function is introduced and some results relating to the C_0 class of contractions introduced by Sz.-Nagy and Foias are generalized.

Let $\mathcal{H}$ be a complex Hilbert space. Recall that an operator T on $\mathcal{H}$ is said to be polynomially bounded if it satisfies the inequality, $\|p(T)\| \leq k\|p\|_\infty$ where p is a polynomial with complex coefficients, $\|p\|_\infty$ denotes the supremum of p on the unit circle and k is a constant (depending upon T). In this note we show that most properties of the C_0 class of contractions (see [4, Chapter 3]) extend naturally to a class of polynomially bounded operators. All of these results would be simple consequences of the conjecture that all polynomially bounded operators are similar to contractions. The fact that it is possible to prove them independently is offered as partial evidence for the validity of this conjecture, at least for this special class of operators.

In what follows A denotes the algebra of complex-valued functions which are continuous on the closed unit disk Δ and holomorphic in $\operatorname{int}\Delta$. The unit circle and the normalized Lebesgue measure on it are denoted by $\partial\Delta$ and m respectively. Let $\phi: A \longrightarrow B(\mathcal{H})$ be the norm-continuous multiplicative homomorphism which extends the homomorphism given by $p \longrightarrow p(T)$, where p is a polynomial.

Definition 1. We say that a polynomially bounded operator T belongs to class D_0 if $\ker\phi \neq (0)$. Note that every T in D_0 determines uniquely a closed set $K \subseteq \partial\Delta$ with $m(K) = 0$ and an inner function F which is analytic off K such that $J = \ker\phi = \{fF, f \in A, f(K) = 0\}$ [2, p. 85]. We adhere to this notation throughout.

Proposition 1. *If $F = 1$, then T is similar to a singular unitary operator.*

Proof. For x, y in $\mathcal{H}$ let $\{\mu[x, y]\}$ denote a finite complex measure on $\partial\Delta$ such that $(p(T)x, y) = \int_{\partial\Delta} p(z)\,d\mu[x, y]$. Since each $\mu[x, y]$ is an annihilating measure for J, $\mu[x, y] = \phi[x, y]m + \nu[x, y]$ where $\phi[x, y] \in \mathcal{H}_0^1$ and

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$\nu[x, y]$ is a singular measure with support $\nu[x, y] \subseteq K$; see [2, p. 86]. Hence $(p(T)x, y) = \int_K p(z)\,d\nu[x, y]$. By Rudin's theorem [2, p. 82] there exists a function g_0 in A such that $\|g_0\| = 1$ and $g_0|K = \bar{z}|K$. Thus $g_0(T) = T^{-1}$ and hence $\|T^{-n}\| = \|g_0^n(T)\| \leq k$. It follows from a theorem of Nagy that T is similar to a unitary, which obviously has singular spectrum.

Remark. Note that once the first part of the proof says that T is singular, the result can be obtained by applying Mlak's more general theorem [4, p. 318].

Proposition 2. *If* $T \in \mathfrak{D}_0$:

(i) $\sigma(T) \cap \{z, |z| < 1\} = \{z, |z| < 1, F(z) = 0\}$.

(ii) $\sigma(T) \cap \partial\Delta \subseteq K$.

(iii) *If F is a Blaschke product, and if Ω is the union of all open arcs across which F is analytic, then* $\partial\Delta \setminus \Omega \subseteq \sigma(T) \cap \partial\Delta$.

Proof. (i) By Fatou's theorem, there exists $g \in A$ such that $\{z, g(z) = 0\} \cap \partial\Delta = K$ [2, p. 80]. If f is the outer part of g and $h = Ff$, then $h \in J$.

Suppose $|\alpha| < 1$ and $F(\alpha) \neq 0$. If $u(z) = (h(z) - h(\alpha))/(z - \alpha)$, $u \in A$ and $u(T)(T - \alpha) = -h(\alpha) \neq 0$. Thus $\alpha \notin \sigma(T)$. Conversely if $|\alpha| < 1$ and $F(\alpha) = 0$, let $h_1(z) = (1 - \bar{\alpha}z)h(z)/(z - \alpha)$. Then $h_1 \in A$ and $(T - \alpha)h_1(T) = 0$. However, since F is the g.c.d. of inner parts of functions in J, $h_1 \notin J$. Thus $\alpha \in \sigma(T)$.

(ii) Suppose $|\alpha| = 1$ and $\alpha \notin K$. Since F is analytic off K, $|F(\alpha)| = 1$. Let h be as in the proof of (i), and let $h_1(z) = (h(z) - h(\alpha))/(z - \alpha)$. Then $(T - \alpha)h_1(T) = -h(\alpha) \neq 0$ implies that $\alpha \notin \sigma(T)$.

(iii) is a consequence of (i).

Definition 2. A polynomially bounded operator T is said to be absolutely continuous (singular) if for each x, y in $\mathcal{H}$, there exists an absolutely continuous (singular) measure $\mu[x, y]$ such that $(p(T)x, y) = \int_{\partial\Delta} p(z)\,d\mu[x, y]$.

Proposition 3. *Suppose T is absolutely continuous. Then,* (i) *ϕ can be extended to a norm-continuous homomorphism* ψ: $\mathcal{H}^\infty \to \mathcal{B}(\mathcal{H})$.

(ii) *If $u \in \mathcal{H}^\infty$ and u is outer, then $u(T)$ is one-one.*

(iii) *If* $\ker \psi \neq (0)$ *and* $m(\sigma(T) \cap \partial\Delta) = 0$, *then* $\ker \phi \neq (0)$.

(iv) *If* $\ker \phi \neq (0)$ *and m_T = g.c.d. of inner parts of nonzero elements of* $\ker \psi$, *then $F = m_T$ = minimum function of T.*

Proof. (i) is a simple consequence of dominated convergence theorem [5, p. 102]. (ii) follows from the fact that if u is outer and $h \in L^1$ are such that $uh \in \mathcal{H}_0^1$ then $h \in \mathcal{H}_0^1$. (iii) Because of (i) it follows, as in the case of contractions, that

$$\sigma(T) \cap \partial\Delta = \partial\Delta \setminus \{\text{union of all arcs on which } m_T \text{ is analytic}\}$$

[5, p. 117]. Hence if $J_0 = \{m_T g,\ g \in A,\ g(\sigma(T) \cap \partial\Delta) = 0\}$, then $J_0 \subseteq A$ and Fatou's lemma ensures that $J_0 \neq (0)$. (iv) Let $J = \ker \phi \neq (0)$. Since there exists an outer function f in A such that fF is in J, $f(T)F(T) = 0$. It follows from (ii) that $F(T) = 0$. Thus $m_T | F$. Hence the set of points on $\partial\Delta$ across which m_T is not analytic has measure zero. Thus by the argument in 3(iii) there is a nonzero outer function g such that $m_T g$ is in J_0. Since it follows easily from Srinivasan's theorem [6, p. 25] that $\ker \psi = m_T \mathcal{H}^\infty$, $m_T(T) = 0$, $J_0 \subseteq J$ and $m_T g = Fh = Fug$ for some h in A with inner-outer factorization ug, where u is inner. Thus $F | m_T$.

Corollary 1. *Suppose T is absolutely continuous, satisfies an $\mathcal{H}^\infty$ function and that the minimum function of T is H. Then there exists a constant M such that for all α with $|\alpha| < 1$ and $H(\alpha) \neq 0$ we have,* $\|(T - \alpha)^{-1}\| \leq M|H(\alpha)|^{-1}/(1 - |\alpha|)$.

Proof. If $|\alpha| < 1$ and $H(\alpha) \neq 0$ then let $G(z) = (H(z) - H(\alpha))/(z - \alpha)$. It follows that G is in $\mathcal{H}^\infty$, $\|G\|_\infty \leq 2/(1 - |\alpha|)$ and $(T - \alpha)^{-1} = H(\alpha)^{-1}G(T)$. Hence

$$\|(T - \alpha)^{-1}\| \leq k\|G\|_\infty |H(\alpha)|^{-1} \leq 2k|H(\alpha)|^{-1}/(1 - |\alpha|).$$

Folk Theorem. *If T is a polynomially bounded operator on $\mathcal{H}$, then there exist unique closed invariant subspaces $\mathcal{H}_a$ and $\mathcal{H}_s$ such that $\mathcal{H} = \mathcal{H}_a \dotplus \mathcal{H}_s$ and if $T_a = T|\mathcal{H}_a$, $T_s = T|\mathcal{H}_s$, then T_a is absolutely continuous and T_s is singular. Specifically, x is in $\mathcal{H}_a$ (or $\mathcal{H}_s$) if and only if for each y in $\mathcal{H}$ there exists a measure $\mu[x, y]$ with $\mu[x, y]$ absolutely continuous (or singular) with respect to m and $(p(T)x, y) = \int p\, d\mu[x, y]$ for all polynomials p.*

It has been brought to our attention that a precise reference for this theorem is [3, Theorem 2.1].

The next theorem gives an explicit description of this decomposition for operators of class $\mathfrak{D}_0$.

Proposition 4. *Suppose T is in $\mathfrak{D}_0$ and let f be an outer function in A such that the set of zeros of f is precisely K. Then $\mathcal{H}_s = \ker f(T)$ and $\mathcal{H}_a =$* closure $R(f(T))$.

Proof. Let $M_0 = \ker f(T)$ and $M_1 =$ closure $R(f(T))$. Then obviously $TM_i \subseteq M_i$ for $i = 0, 1$. If x is in M_0, y is in $\mathcal{H}$ and $\mu[x, y]$ is such that $(p(T)x, y) = \int p(z)\, d\mu[x, y]$, then by F. and M. Riesz theorem $f\mu[x, y]$ is absolutely continuous. Thus if $\mu[x, y] = h[x, y]m + \nu[x, y]$ is the Lebesgue decomposition, it follows easily that $h[x, y]$ is in $\mathcal{H}_0^1$ and $\mu[x, y]$ can be replaced by $\nu[x, y]$. Hence $M_0 \subseteq \mathcal{H}_s$. Conversely suppose x is in $\mathcal{H}_s$. If y is in $\mathcal{H}$, there exists a unique singular measure $\nu[x, y]$ such that $(p(T)x, y) =$

$\int_{\partial\Delta}p(z)d\nu[x, y]$. (Uniqueness is a simple consequence of the F. and M. Riesz theorem.) Since $\nu[x, y]$ annihilates J, support $\nu[x, y] \subseteq K$ [2, p. 86]. Hence $(f(T)x, y) = \int_K f(z)d\nu[x, y] = 0$ since $f = 0$ on K. Since y is arbitrary, $f(T)x = 0$. Thus $\mathcal{H}_s = M_0$. Since $T_aT_s = 0 = T_sT_a$, we have $f(T) = f(T_a) + f(T_s) = f(T_a)$. Thus $M_1 = \overline{f(T_a)\mathcal{H}_a}$. Since T_a is absolutely continuous so is T_a^* and since f is outer so is $\tilde{f}(z) = \overline{f(\overline{z})}$. So by Proposition 3(ii) $f(T_a)^* = \tilde{f}(T_a^*)$ has zero kernel and $\overline{f(T_a)\mathcal{H}_a} = \mathcal{H}_a$.

From now on we assume that T is absolutely continuous and satisfies an $\mathcal{H}^\infty$ function. We denote the minimum function of T by F and the Fourier coefficients of F by $\{c_n\}_{n=0}^\infty$. Note that for any absolutely continuous operator T, $T^n \to 0$ weakly. (This is an immediate corollary of the Riemann-Lebesgue lemma.) We prove that if such an operator also satisfies an $\mathcal{H}^\infty$ function then the convergence takes place in the strong topology.

Proposition 5. *If F is a finite Blaschke product, then $\|T^n\| \to 0$.*

Proof. Since F is analytic across all of $\partial\Delta$, by [5, Theorem 5, p. 126] we have $\sigma(T) \cap \partial\Delta = \emptyset$. Thus the spectral radius of T is less than 1 and hence $\|T^n\| \to 0$.

Proposition 6. *Suppose that $F(0) \neq 0$. Then $T^n \to 0$ strongly.*

Proof. Let $M = \{h \in \mathcal{H}, \|T^nh\| \to 0\}$. Note that since T is power-bounded, M is closed. Let P be the orthogonal projection on M and let $P^\perp = I - P$. If $S = P^\perp T|M^\perp$, then $S^n = P^\perp T^n|M^\perp$ since $TM \subseteq M$. Hence S is polynomially bounded, absolutely continuous and moreover $F(S) = 0$. If H is the minimum function of S then $H|F$ and hence it follows from Proposition 2(i) that $0 \notin \sigma(S)$ and in particular $\ker S^* = (0)$. Next if $h \in M^\perp$ and $\|S^nh\| = \|P^\perp T^nh\| \to 0$, then we claim that $h = 0$. Now for $m \geq 0$,

$$\|T^{m+n_0}h\| = \|T^m(PT^{n_0}h + P^\perp T^{n_0}h)\| \leq \|T^m(PT^{n_0}h)\| + k\|P^\perp T^{n_0}h\|.$$

The second term can be chosen to be arbitrarily small and $PT^{n_0}h \in M$. Thus $\|T^{m+n_0}h\| \to 0$ and $h \in M$. Thus $h = 0$. Now applying the first part of Proposition 5.3 of [5, p. 72] to S on $M^\perp$, we get the existence of a unitary U and a one-one bounded operator X with dense range such that $XS = UX$. By the F. and M. Riesz theorem the spectral measure of U is absolutely continuous and hence $H(U)$ has no null-space. However, we also have $H(U)X = XH(S) = 0$ and since range of X is dense, it follows that $M^\perp = (0)$. Hence $M = \mathcal{H}$.

Corollary 2. *If T is absolutely continuous and satisfies an $\mathcal{H}^\infty$ function i.e. if $\ker \psi \neq (0)$ then $T^n \to 0$ strongly and $T^{*n} \to 0$ strongly.*

Proof. Let $F = z^n Q$ where $Q(0) \neq 0$. If $\mathcal{H}_1 = \{h \in \mathcal{H}, Th = 0\}$ and $\mathcal{H}_2 = \{h \in \mathcal{H}, Q(T)h = 0\}$ then the proof of Theorem 6.3 in [5, Chapter III] yields that $\mathcal{H}_1 \vee \mathcal{H}_2 = \mathcal{H}$. Since Proposition 5 applies to $T|\mathcal{H}_1$ and Proposition 6 to $T|\mathcal{H}_2$ we have $T^n \xrightarrow{st} 0$.

A particular case $(k = 1)$ of Corollary 2 yields the following well-known theorem proved by Sz.-Nagy and Foiaş [5, p. 114].

Corollary 3. $C_0 \subseteq C_{00}$.

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