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A FOUR-VERTICES THEOREM FOR RULED SURFACES

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ABSTRACT. The skewness of distribution of an orientable closed ruled surface satisfies a four-vertices theorem.

The concept of the skewness of distribution μ of a generator of a ruled surface was introduced by V. Rangachariar [1]. It is well known that μ is an invariant in the Euclidean geometry of ruled surfaces. In this paper, it is shown that μ has at least four extrema when it is associated to a certain curve on a unit sphere.

An oriented straight line in E^3 can be represented by a dual vector

$$\mathbf{A} = \mathbf{a} + r\tilde{\mathbf{a}}$$

where a is a unit vector along the straight line, $\tilde{\mathbf{a}}$ is the moment of a about the origin of coordinates O and r is an indeterminate subject to the relation $r^2 = 0$. By the definitions, $\mathbf{a}^2 = 1$, $\mathbf{a} \cdot \tilde{\mathbf{a}} = 0$.

A ruled surface is represented by a curve on the dual unit sphere [2, §8-2]

$$\mathbf{A}_1 = \mathbf{A}_1(u) = \mathbf{a}_1(u) + r\tilde{\mathbf{a}}_1(u), \qquad u_0 \le u \le u_1,$$

where u is a real parameter. The $\mathbf{A}(u)$ are identified with the generators of the surface. A closed ruled surface is represented on the dual unit sphere either by a closed curve or by a curve for which $\mathbf{A}(u_1) = -\mathbf{A}(u_0)$ since $\mathbf{A}$ and $-\mathbf{A}$ represent the same line. In the first case, we say that the ruled surface is *orientable*, in the second case *nonorientable*.

Let S be a smooth, ruled surface generated by $\mathbf{A}_1$. With each generator $\mathbf{A}_1$ we associate an orthonormal trihedron determined by the generator, the normal to the ruled surface and the tangent to the ruled surface at the central point of the generator. Let $\mathbf{A}_1, \mathbf{A}_2, \mathbf{A}_3$ be the dual vectors of the three edges of this trihedron. Then [2, p. 166]

$$\mathbf{A}_1' = K\mathbf{A}_2, \qquad \mathbf{A}_2' = -K\mathbf{A}_1 + T\mathbf{A}_3, \qquad \mathbf{A}_3' = -T\mathbf{A}_2,$$

where primes denote differentiation with respect to u. We follow the notation of [2] and put

$$K = k_1 + rk_2, \qquad T = t_1 + rt_2;$$

then

$$\mathbf{a}_1' = k_1\mathbf{a}_2, \qquad \mathbf{a}_2' = -k_1\mathbf{a}_1 + t_1\mathbf{a}_3, \qquad \mathbf{a}_3' = -t_1\mathbf{a}_2.$$

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Also,

$$\mu = \det(\mathbf{a}_1, \mathbf{a}_1', \mathbf{a}_1'') \cdot |\mathbf{a}_1'|^{-3} = t_1/k_1.$$

Let S be an orientable, closed, ruled surface.

Then $\mathbf{A}_i(u_1) = \mathbf{A}_i(u_0)$, $i = 1, 2, 3$, and

$$\oint_S \mathbf{a}_1 \, d\mu = \int_{u_0}^{u_1} \mathbf{a}_1 \, d(t_1/k_1) = -\int_{u_0}^{u_1} (t_1/k_1) \, d\mathbf{a}_1 = -\int_{u_0}^{u_1} t_1 \mathbf{a}_2 \, du = \oint_S d\mathbf{a}_3 = 0.$$

A curve on the surface of a sphere is called *spherically convex* if (a) it does not contain a pair of antipodal points and (b) it is the intersection of the sphere and a convex cone whose apex is the centre O of the sphere.

Let the vector $\mathbf{a}_1(u)$ generate a spherically convex closed curve C on the unit sphere with centre O. We will show now that the function μ has at least four extrema on C. As it is continuous it has extreme values. We will show following [3] that it cannot have only two extrema.

Let m and M be the points on C where this function has minimum and maximum values respectively. Let the function be increasing along one arc and decreasing on the other arc joining m and M. Therefore, along one arc, $d\mu > 0$ and along the other, $d\mu < 0$. The equation of the plane OmM is $\mathbf{a}_1 \cdot \mathbf{n} = 0$ where $\mathbf{n}$ is a constant vector. Hence $\mathbf{a}_1 \cdot \mathbf{n} > 0$ on one side of the plane and $\mathbf{a}_1 \cdot \mathbf{n} < 0$ on its other side. Therefore, let us say that $\mathbf{a}_1 \cdot \mathbf{n} > 0$ at points on the open arc $d\mu > 0$ and $\mathbf{a}_1 \cdot \mathbf{n} < 0$ at points on the open arc $d\mu < 0$.

Hence we find that $(\mathbf{a}_1 \cdot \mathbf{n})d\mu > 0$ for points on C on both sides of the plane OmM and consequently $\oint_S (\mathbf{a}_1 \cdot \mathbf{n}) d\mu > 0$. But

$$\int_S (\mathbf{a}_1 \cdot \mathbf{n}) \, d\mu = \left[\oint_S \mathbf{a}_1 \, d\mu\right] \cdot \mathbf{n} = 0$$

as $\mathbf{n}$ is a constant vector.

Thus we have a contradiction. Hence the function μ has more than two extrema. We have proved the following theorem:

If the unit vectors along the generators of a C^2, orientable ruled surface induce a spherically convex closed curve C on the unit sphere, then the skewness of distribution has at least four extrema on C.

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