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ON BIESTERFELDT'S COMPLETION AXIOM SPACES

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ABSTRACT. It is proved that a Hausdorff, totally bounded Completion Axiom space is a uniform space. The method of proof shows that Hausdorff Completion Axiom spaces have completions (in the embedding sense) which are again Hausdorff Completion Axiom spaces; moreover these completions are uniformly regular, uniformly strict, and have the regular extension property.

1. **Introduction.** The first uniform convergence spaces [2] to be completed, in the embedding sense, were the Completion Axiom (hereafter abbreviated C.A.) spaces of Biesterfeldt [1]. Later Richardson showed that indeed any uniform convergence space has a completion [7]; and since then various papers on completions of uniform convergence spaces or their Cauchy structures have appeared. (See, for example [4] or [5].)

However, the extent to which these original C.A. spaces deviate from uniform spaces is still not determined. The only known result is that a C.A. space is still not determined. The only known result is that a C.A. space has a regular topology as its induced convergence structure [6]. In this note it will be shown that a Hausdorff, totally bounded C.A. space is a uniform space. In doing so, we will prove that Hausdorff C.A. spaces have very strong (in a sense to be made precise below) completions, with extension property, within their own category.

The reader is referred to [2] and [3] for basic definitions although a few pertinent ones are given here. If (X, I) is a uniform convergence space (u.c.s.), a Δ-symmetric base for I is a base consisting of symmetric filters, each of which is coarser than the diagonal filter. Then (X, I) is a C.A. space if there exists a Δ-symmetric base β for I such that $\mathfrak{F}$ is Cauchy in X iff $\mathfrak{F} \times \mathfrak{F} \geq \Phi$ for every $\Phi \in \beta$. (X, I) is *uniformly regular* if $\mathrm{cl}_{X \times X}\, \Phi \in I$ whenever $\Phi \in I$. Completions are taken in the embedding sense; that is (X^*, I^*) is a completion of (X, I) if there is a uniform homeomorphism of (X, I) onto a dense subspace of (X^*, I^*) and (X^*, I^*) is complete. A completion (X^*, I^*) of (X, I) is *uniformly strict* if there is some $\Phi^* \in I^*$ such that when $V^* \in \Phi^*$ and $y \in X^*$, $V^*(y) \cap X \neq \varnothing$.

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2. Results.

Proposition 1. *Let (X, I) be a u.c.s. with Δ-symmetric base β. Then, for $A \subset X \times X$, $\mathrm{cl}_{X \times X} A = \bigcup \bigcap (V \circ A \circ V: V \in \Phi)$, the union being taken over all $\Phi \in \beta$.*

Proof. Notice first that for any u.c.s. (X, I) with Δ-symmetric base β and $B \subset X$, $\mathrm{cl}_X B = \bigcup (B(\Phi): \Phi \in \beta)$ where $B(\Phi) = \{x: V(x) \cap B \neq \emptyset$ for all $V \in \Phi\}$.

Now let β^* be any Δ-symmetric base for the product u.c.s. and let $(x, y) \in \mathrm{cl}_{X \times X} A$. By applying the remark above to β^*, it follows that $(x, y) \in A(\Phi^*)$ for some $\Phi^* \in \beta^*$. If p_1, p_2 are the projection maps it is seen that $(p_1 \times p_1)\Phi^* \circ (p_2 \times p_2)\Phi^*$ exists and is therefore finer than some $\Psi \in \beta$. We assert that $(x, y) \in \bigcap (V \circ A \circ V: V \in \Psi)$. For if $V \in \Psi$ and $(p_1 \times p_1)\mathbf{V} \circ (p_2 \times p_2)\mathbf{V} \subset V$, $\mathbf{V}$ symmetric in Φ^*, then $\mathbf{V}(x, y) \cap A$ is not empty so there exists $(a_1, a_2) \in A$ with $((a_1, a_2), (x, y)) \in \mathbf{V}$. From this, $(x, y) \in (p_1 \times p_1)\mathbf{V} \circ A \circ (p_2 \times p_2)\mathbf{V} \subset V \circ A \circ V$.

Conversely, assume $(x, y) \in \bigcap (V \circ A \circ V: V \in \Phi)$ for some $\Phi \in \beta$. Then there are points $(x, a(V)) \in V$, $(a(V), b(V)) \in A$, $(b(V), y) \in V$. The filters of sections of the nets $(a(V): V \in \Phi)$, $(b(V): V \in \Phi)$, converge, respectively, to x, y. Hence the filter of sections of the net $((a(V), b(V)): V \in \Phi)$ converges to (x, y) and contains A. This means $(x, y) \in \mathrm{cl}_{X \times X} A$.

Proposition 2. *Let (X, I) have a uniformly regular, uniformly strict completion (X^*, I^*). Then I^* is generated by filters of the form $\mathrm{cl}_{X^* \times X^*} \Phi$, $\Phi \in I$.*

Proof. First, if $\Phi \in I$, $\mathrm{cl}_{X^* \times X^*} \Phi \in I^*$ follows from the uniform regularity of (X^*, I^*).

Let $\Phi^* \in I^*$ and put $\Phi_1^* = \Phi^* \wedge \Psi^*$ where $V^*(x) \cap X \neq \emptyset$ for all $x \in X^*$, $V^* \in \Psi^*$. With no loss of generality assume Φ_1^* is symmetric and coarser than the diagonal filter. Define $\Sigma^* = \Phi_1^* \circ \Phi_1^* \circ \Phi_1^*$; we will show that $\Phi^* \geq \mathrm{cl}_{X^* \times X^*} \Sigma$, where Σ is the restriction of Σ^* to $X \times X$. Let $C \in \Sigma$, $C = C^* \cap X \times X$, $C^* \in \Sigma^*$. Choose a symmetric $V_1^* \in \Phi_1^*$ such that $V_1^* \circ V_1^* \circ V_1^* \subset C^*$. Let $(x, y) \in V_1^*$. To show that $(x, y) \in \mathrm{cl}_{X^* \times X^*} C$, it suffices, by Proposition 1, to prove that $(x, y) \in \bigcap (V^* \circ C \circ V^*: V^* \in \Phi_1^*, V^*$ symmetric, $V^* \subset V_1^*)$. Let V^* be as indicated; then $V^* \in \Psi^*$ so there are points $z_1 \in V^*(x) \cap X$, $z_2 \in V^*(y) \cap X$. Thus $(z_1, z_2) \in V^* \circ V_1^* \circ V^* \subset C^*$. Also $(z_1, z_2) \in C$ because $z_1, z_2 \in X$. Hence $(x, y) \in V^* \circ C \circ V^*$ as desired.

Proposition 3. *Let (X, I) be uniformly regular with uniformly strict, uniformly regular completion (X^*, I^*). If (Y, D) is any complete, uniformly regular u.c.s. and $f: (X, I) \to (Y, D)$ is uniformly continuous, then f has a uniformly continuous extension $f^*: (X^*, I^*) \to (Y, D)$.*

Proof. Define f^* by $f^*(y) = \lim f(\mathfrak{F})$ where $\mathfrak{F} \to y$ in X^* and $X \in \mathfrak{F}$. Since the embedding $X \to X^*$ is also an embedding of the Cauchy structures, f^* is well defined. Let $\Phi^* \in I^*$. By Proposition 2, $\Phi^* \geq \mathrm{cl}_{X^* \times X^*} \Phi$ for some $\Phi \in I$. Since (Y, D) is uniformly regular, $\mathrm{cl}_{Y \times Y}(f \times f)\Phi \in D$, so to prove that f^* is uniformly continuous it suffices to show that $(f^* \times f^*)(\mathrm{cl}_{X^* \times X^*} \Phi) \geq \mathrm{cl}_{Y \times Y}(f \times f)\Phi$. We claim, in fact, that, if $A \in \Phi$, $(f^* \times f^*)(\mathrm{cl}_{X^* \times X^*} A) \subset \mathrm{cl}_{Y \times Y}(f \times f)A$. For if $(x, y) \in \mathrm{cl}_{X^* \times X^*} A$, then $(x, y) \in \bigcap (V^* \circ A \circ V^*: V^* \in \Phi^*)$ for some $\Phi^* \in I^*$ by Proposition 1. As in the last part of the proof of Proposition 1, there are nets $a(V^*) \to x$, $b(V^*) \to y$ with the section filter $\mathfrak{G}$ of the product net containing A. Then

$$(f^* \times f^*)(x, y) = \lim (f \times f)\mathfrak{G} \in \mathrm{cl}_{Y \times Y}(f \times f)A.$$

Remark. The referee has made the observation that, for C.A. spaces, the completion of Theorem 1 below is that of Biesterfeldt [1] when equivalent Cauchy filters are identified. Moreover, he observes that if $\mathfrak{U}(\langle \mathfrak{F} \rangle)$ is chosen to be an ultrafilter in $\langle \mathfrak{F} \rangle$, then the completion of Theorem 1 is the Kowalsky completion. (See [5, p. 103].)

Theorem 1. *Let (X, I) be a Hausdorff C.A. space. Then (X, I) has a uniformly regular, uniformly strict, Hausdorff completion which also is a C.A. space. This completion has the extension property relative to uniformly continuous maps and complete, uniformly regular u.c.s.*

Proof. Let $X^\wedge$ be the collection of equivalence classes $\langle \mathfrak{F} \rangle$ of Cauchy filters on X under the relation $\mathfrak{F} \sim \mathfrak{G}$ if $\mathfrak{F} \times \mathfrak{G}$ is in I. For each $\langle \mathfrak{F} \rangle$ let $\mathfrak{U}(\langle \mathfrak{F} \rangle)$ be an arbitrary Cauchy filter equivalent to $\mathfrak{F}$ with the understanding that $\mathfrak{U}(\langle \dot{x} \rangle) = \dot{x}$. For $S \subset X \times X$ let $S^\wedge$ be the collection of all pairs $(\langle \mathfrak{F} \rangle, \langle \mathfrak{G} \rangle)$ such that $A \times B \subset S$ for some A, B in $\mathfrak{U}(\langle \mathfrak{F} \rangle)$, $\mathfrak{U}(\langle \mathfrak{G} \rangle)$ respectively. Let $\boldsymbol{\beta}$ be an admissible Δ-symmetric base for the C.A. space (X, I). For each $\Phi \in \boldsymbol{\beta}$ define $\Phi^\wedge$ to be the filter generated by the $A^\wedge$, $A \in \Phi$. If $A \in \Phi$, $\Phi \in \boldsymbol{\beta}$ and $\mathfrak{F}$ is any Cauchy filter on X, then $\mathfrak{F} \times \mathfrak{F} \geq \Phi$ by C.A. Hence $(\langle \mathfrak{F} \rangle, \langle \mathfrak{F} \rangle) \in A^\wedge$ and each $\Phi^\wedge$ is coarser than the diagonal filter in $X^\wedge \times X^\wedge$. Standard arguments show, then, that the collection $\boldsymbol{\beta}^\wedge$ of all $\Phi^\wedge$, $\Phi \in \boldsymbol{\beta}$ is a Δ-symmetric base for a u.c.s. $I^\wedge$ on $X^\wedge$. Define $j: X \to X^\wedge$ by $j(x) = \langle \dot{x} \rangle$. Notice that if $\Phi \in \boldsymbol{\beta}$ and $\langle \mathfrak{F} \rangle \in X^\wedge$, $A \in \Phi$, then $\mathfrak{U}(\langle \mathfrak{F} \rangle) \times \mathfrak{U}(\langle \mathfrak{F} \rangle) \geq \Phi$ by C.A. so $U \times U \subset A$ for some $U \in \mathfrak{U}(\langle \mathfrak{F} \rangle)$. Thus whenever $y \in U$, $\langle \dot{y} \rangle \in A^\wedge(\langle \mathfrak{F} \rangle)$. This shows that $(X^\wedge, I^\wedge)$ is uniformly strict if, in fact, it is a completion. $(X^\wedge, I^\wedge)$ is complete: Suppose $\Omega \times \Omega \geq \Phi^\wedge$ for $\Phi \in \boldsymbol{\beta}$ and define, for $T \in \Omega$, $A \in \Phi$, $[A, T] = \{x: \langle \dot{x} \rangle \in A^\wedge(T)\}$. The preceding paragraph shows that $[A, T] \neq \emptyset$ and it follows easily that the collection $\mathfrak{F}$ of supersets of the $[A, T]$ is a filter on X. In fact $\mathfrak{F}$ is a Cauchy filter. For if $A \in \Phi$ there exists $T \in \Omega$ such that $T \times T \subset A^\wedge$. If $(x, y) \in [A, T] \times [A, T]$ then there are filters $\mathfrak{G}$, $\mathfrak{H}$ in T such that $(\langle \dot{x} \rangle, \langle \mathfrak{G} \rangle) \in A^\wedge$, $(\langle \dot{y} \rangle, \langle \mathfrak{H} \rangle) \in A^\wedge$. Hence $(\langle \dot{x} \rangle, \langle \dot{y} \rangle) \in A^\wedge \circ$

$A^\wedge \circ A^\wedge \subset (A \circ A \circ A)^\wedge$ or $(x, y) \in A^3$. This means $\mathfrak{F} \times \mathfrak{F} \geq \Phi^3$ so $\mathfrak{F}$ is Cauchy. It is now asserted that $\Omega \to \langle \dot{\mathfrak{F}} \rangle$, in fact that $\Omega \times \langle \dot{\mathfrak{F}} \rangle \geq \Phi^\wedge$. If $A \in \Phi$ then $\mathfrak{F} \times \mathfrak{F} \geq \Phi$ by C.A. so $[A_1, T] \times [A_1, T] \subset A$ for some $T \in \Omega$, $A_1 \in \Phi$, $A_1 \subset A$. Let $\langle \mathfrak{G} \rangle \in T$; using C.A. again, $U \times U \subset A_1$ for some $U \in \mathfrak{U}(\langle \mathfrak{G} \rangle)$ hence $U \subset (x: \langle \dot{x} \rangle \in A_1^\wedge(\langle \mathfrak{G} \rangle) \subset [A_1, T]$. Thus $U \times [A_1, T] \subset A$, hence $(\langle \mathfrak{G} \rangle, \langle \mathfrak{F} \rangle) \in A^\wedge$. That is, $T \times \{\langle \mathfrak{F} \rangle\} \subset A^\wedge$.

The fact that j is a uniform homeomorphism follows from the relation $(j \times j)\Phi \geq \Phi^\wedge$ and the fact that $(j \times j)\Psi \geq \Phi^\wedge$ implies $\Psi \geq \Phi$. It is obvious that $(X^\wedge, I^\wedge)$ is T_1, hence Hausdorff.

Notice that any C. A. space (X, I) is uniformly regular because $\mathrm{cl}_{X \times X} \Phi \geq \Phi^3$ by Proposition 2 of [6]. Thus the proof of this theorem is complete if it can be shown that $(X^\wedge, I^\wedge)$ is a C.A. space. For this suppose Ω is Cauchy in $X^\wedge$, $\Omega \times \Omega \geq \Phi^\wedge$, and let Φ_1 be any member of $\boldsymbol{\beta}$. The $\mathfrak{F}$ constructed in the proof of completeness is Cauchy in X so, by C.A. in X, $\mathfrak{F} \times \mathfrak{F} \geq \Phi_1$. Hence, to show $\Omega \times \Omega \geq \Phi_1^\wedge$ it suffices to prove $\Omega \times \Omega \geq (\mathfrak{F} \times \mathfrak{F})^\wedge$. For this let $[A, T]$ be given in $\mathfrak{F}$; we may assume $T_1 \times T_1 \subset A^\wedge$ for some $T_1 \in \Omega$, $T_1 \subset T$. Then $T_1 \times T_1 \subset ([A, T_1] \times [A, T_1])^\wedge$, for if $\langle \mathfrak{G} \rangle$, $\langle \mathfrak{H} \rangle \in T_1$, there are $U \in \mathfrak{U}(\langle \mathfrak{G} \rangle)$, $V \in \mathfrak{U}(\langle \mathfrak{H} \rangle)$ such that $U \times U \subset A$, $V \times V \subset A$ (C.A. has been used again.) so $U \subset (x: \langle \dot{x} \rangle \in A^\wedge(\langle \mathfrak{G} \rangle)) \subset [A, T_1]$ and $V \subset (x: \langle \dot{x} \rangle \in A^\wedge(\langle \mathfrak{H} \rangle) \subset [A, T_1]$, thus $(\langle \mathfrak{G} \rangle, \langle \mathfrak{H} \rangle) \in ([A, T_1] \times [A, T_1])^\wedge$.

We note that the extension property of $(X^\wedge, I^\wedge)$ and the fact that a uniform space is uniformly regular implies that $(X^\wedge, I^\wedge)$ is the usual completion of (X, I) whenever I is a Hausdorff uniformity.

Proposition 4. *Let (X, I) be a C.A. space. Then (X, I) is totally bounded if and only if there exists $\Phi \in I$ such that whenever $V \in \Phi$, there is a finite subset F of X such that $V(F) = X$.*

Proof. Suppose (X, I) is totally bounded and let Φ be a member of an admissible Δ-symmetric base for I. Suppose for some $V \in \Phi$, $\{X - V(F)$: F is finite$\}$ is a filter base on X. If this filter base is coarser than a Cauchy filter $\mathfrak{F}$, then $\mathfrak{F} \times \mathfrak{F} \geq \Phi$ by C.A. A contradiction now is obtained exactly like the uniform space case. Conversely suppose the condition of the proposition holds and let $\mathfrak{U}$ be an ultrafilter on X. If $V \in \Phi$ is symmetric, some $V(x) \in \mathfrak{U}$ and $V(x) \times V(x) \subset V^2$, so $\mathfrak{U} \times \mathfrak{U} \geq \Phi^2$ and $\mathfrak{U}$ is Cauchy.

Theorem 2. *A totally bounded Hausdorff C.A. space is a uniform space.*

Proof. Let $(X^\wedge, I^\wedge)$ be the completion of Theorem 1. By the proof of strictness of $(X^\wedge, I^\wedge)$, $\Phi \in \boldsymbol{\beta}$ implies $V^\wedge(y) \cap j(X) \neq \varnothing$ for each $y \in X^\wedge$, $V \in \Phi$. Hence, by Proposition 4, total boundedness of (X, I) carries over to $(X^\wedge, I^\wedge)$. So the structure induced by $I^\wedge$ is a compact Hausdorff topology. But there is only one possible C.A. structure which induces a compact, Hausdorff topology, namely a uniformity [6, Proposition 2]. Since (X, I) is embedded uniformly in $(X^{\wedge}, I^{\wedge})$, (X, I) is a uniform space.

Corollary 1. *Let (X, I) be a Hausdorff C.A. space. Then I induces a completely regular topology on X if and only if there is a totally bounded, Hausdorff C.A. structure on X which induces the same topology as I.*

Whether each C.A. space is a uniform space remains unresolved. A counterexample would produce a proper class of u.c.s. strictly larger than the class of uniform spaces which has completions within its own category.

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