$R_3$-QUASI-UNIFORM SPACES
AND TOPOLOGICAL HOMEOMORPHISM GROUPS

MASSOOD SEYEDIN

ABSTRACT. It is well known that if $X$ is a completely regular space and $G$ is a homeomorphism group of $X$ onto itself such that $G$ is equicontinuous with respect to a compatible uniformity of $X$, then $G$ is a topological group under the topology of pointwise convergence. In this paper, we obtain a generalization of the above result by means of $R_3$-quasi-uniformities.

1. Introduction. Let $(X, r)$ be a topological space. It is well known that if $\mathcal{U}$ is a compatible uniformity on $X$ such that $G$ is a homeomorphism group that is equicontinuous with respect to $\mathcal{U}$, then $G$ is a topological group under the topology of pointwise convergence. R. V. Fuller has obtained an analogous result for regular spaces [2] and we have shown previously that a similar result applies when $(X, r)$ is only an $R_0$ space (and hence, in particular, if $X$ is $T_1$ or regular) [6]. In this paper we use $R_3$-quasi-uniformities to complement Fuller’s result. We take the domain space $(X, r)$ to be an arbitrary topological space. If our domain space is regular, it is known that there exists a compatible $R_3$-quasi-uniformity $\mathcal{U}$ on $X$, that is, $r = r_\mathcal{U}$ [5]. Finally we give a simple example of a non-$R_0$ topological space (hence not regular) for which our principal result, Theorem 6, obtains.

Let $Y$ be a topological space. A collection $\mathcal{O}^*$ of two-element open covers of $Y$ is said to be a semiuniformity for $Y$ if for each $q \in Y$ and each neighborhood $V$ of $q$ there is $\{V_1, V_2\} \in \mathcal{O}^*$ such that $q \in V_1 \subset V$ and $Y - V_2$ is a neighborhood of $q$ [2]. Let $F$ be a family of functions from a topological space $X$ to semiuniform space $(Y, \mathcal{O}^*)$. Then $F$ is semiequicontinuous if for each $V \in \mathcal{O}^*$ there is an open cover $\mathcal{A}$ of $X$ such that $\mathcal{A}$ refines $f^{-1}(V)$ for each $f \in F$ [2]. One may easily show that a topological space has a semiuniformity if and only if it is regular.

Let $X$ be a nonempty set. A quasi-uniformity for $X$ is a filter $\mathcal{U}$ of reflexive subsets of $X \times X$ such that if $U \in \mathcal{U}$, there is $V \in \mathcal{U}$ such that $V \circ V \subset U$ [5]. Let $G$ be a collection of maps from a topological space $(X, r)$ into a quasi-uniform space $(Y, \mathcal{U})$ and let $x \in X$. Then $F$ is quasi-equicontinuous at $x$ provided that for each $U \in \mathcal{U}$ there exists a neighbor-
hood $N$ of $x$ such that for $f \in F$, $f(N) \subseteq U(f(x))$ and $F$ is quasi-equicontinuous provided $F$ is quasi-equicontinuous at each $x \in X$. If $y \in Y$ and $U_1 \in \mathcal{U}$ such that $U_1(y)$ is open and $U_2 \in \mathcal{U}$ such that $U_2 \circ U_1 \circ U_2 \circ U_2(y) \subseteq U_1(y)$ and $U_2 = U_2^{-1}$, then $I = \{U_1(y), \bigcup \text{int } U_2(p) ; p \notin U_2 \circ U_2(y)\}$ is a two element quasi-uniform cover of $X$. A quasi-uniform space $(X, \mathcal{U})$ is $R_3$, if, given $x \in X$ and $U \in \mathcal{U}$, there exists a symmetric $W \in \mathcal{U}$ such that $W \circ W(x) \subseteq U(x)$ [3]. It is shown that if $(X, r)$ is regular, then the Pervin quasi-uniformity on $X$ is $R_3$ [5, Theorem 3.17].

2. Topological groups of homeomorphisms.

Theorem 1. Let $(Y, \mathcal{U})$ be an $R_3$-quasi-uniform space. Then the collection of all two element quasi-uniform covers of $Y$ is a semiuniformity for $Y$.

Proof. Let $q \in Y$ and let $V$ be a neighborhood of $q$. Let $U_1 \in \mathcal{U}$ such that $U_1(q) \subseteq V$ and $U_1(q)$ is open. By hypothesis there is a symmetric entourage $U_2 \in \mathcal{U}$ such that $U_2 \circ U_2 \circ U_2 \circ U_2(q) \subseteq U_1(q)$. Let $C = \{U_1(q), \bigcup \text{int } U_2(y) ; y \notin U_2 \circ U_2(q)\}$. Suppose that $x \in Y$ and $x \notin U_1(q)$. Note that if $z \in Y$ and $z \in U_2 \circ U_2(q)$, then $U_2(z) \subseteq U_1(q)$. Thus $x \notin U_2 \circ U_2(q)$ and $x \in \text{int } U_2(x)$. Therefore $C$ is an open cover of $Y$. Furthermore, let $p \in U_2(q)$ and suppose that $p \in V_2 = \bigcup \text{int } U_2(y) ; y \notin U_2 \circ U_2(q)$. Then there exists a $y \in Y$ such that $y \in U_2(q)$ and $y \notin U_2 \circ U_2(q)$. But $y \in U_2(p) \subseteq U_2 \circ U_2(q)$—a contradiction. Then $\mathcal{U}^* = \{C ; q \in Y \text{ and } V \text{ is a neighborhood of } q\}$ is a semiuniformity for $Y$.

The semiuniformity $\mathcal{U}^*$ of the preceding theorem will be called a quasi-uniform semiuniformity.

Theorem 2. Let $(Y, \mathcal{U})$ be an $R_3$-quasi-uniform space and let $F$ be a family of quasi-equicontinuous functions from a topological space $(X, r)$ into $(Y, \mathcal{U})$. Then $F$ is semiequicontinuous with respect to the quasi-uniform semiuniformity of $\mathcal{U}$.

Proof. Let $\mathcal{U}^*$ be the quasi-uniform semiuniformity of $\mathcal{U}$, let $y, q \in Y$ and $U_1, U_2 \in \mathcal{U}$. Let $l \in \mathcal{U}^*$ such that $l = \{U_1(q), \bigcup \text{int } U_2(y) ; y \notin U_2 \circ U_2(q)\}$. By hypothesis, for each $x \in X$ there exists a neighborhood $N_x$ of $x$ such that for all $f \in F$, $f(N_x) \subseteq U_2(f(x))$. It may be seen that $U_2(f(x))$ is contained in either $U_1(q)$ or $V_2 = \bigcup \text{int } U_2(y) ; y \notin U_2 \circ U_2(q)$ as follows: Let $z_1, z_2 \in U_2(f(x))$, so that $z_1 \notin U_1(q)$ and $z_2 \notin V_2$. Now if $z_2 \notin V_2$, then $z_2 \in U_2 \circ U_2(q)$ and $(q, z_2) \in U_2 \circ U_2$. Since $(z_2, f(x)) \in U_2$ and $(f(x), z_1) \in U_2$, $z_1 \in U_2 \circ U_2 \circ U_2(q) \subseteq U_1(q)$—a contradiction. Thus $\{N_x ; x \in X\}$ is the desired open cover of $X$.

The proof of the following theorem is based on the proof of [2, Theorem 4].

Theorem 3. Let $F$ be a family of one-to-one functions of a topological space $(X, r)$ onto itself. Let $\mathcal{U}$ be an $R_3$-quasi-uniformity on $X$ such that
If $F^{-1}$ is $\mathcal{U}$-quasi-equicontinuous, then the mapping $\Psi : F \to F$, defined by $\Psi(f) = f^{-1}$, is continuous relative to the topology of pointwise convergence on $F$ and $F^{-1}$.

**Proof.** Throughout the proof, if $p \in X$ and $U \in t$, then $W(p, U)$ denotes \{ $f \in F$: $f(p) \in U$ \}. Let $\mathcal{U}^*$ be the quasi-uniform semiuniformity of $\mathcal{U}$. Let $g \in F$, $p \in X$ and $V \in t$ such that $W(p, V)$ is a neighborhood of $g$. Since $t \subseteq t_0$ there is $\{ V_1, V_2 \} \in \mathcal{U}^*$ such that $g^{-1}(p) \subseteq V_1 \subseteq V$ and $X - V_2$ is a $t_0$ neighborhood of $g^{-1}(p)$. By Theorem 2, $F^{-1}$ is semiequicontinuous with respect to $\mathcal{U}^*$. Let $U$ be a $t$-open cover of $X$ such that $U$ refines $\{ f(V_1), f(V_2) \}$ for all $f \in F$ and let $U$ be a member of $\mathcal{U}$ that contains $p$. Then $W(g^{-1}(p), U)$ is a neighborhood of $g$. Let $f \in F$ such that $f \in W(g^{-1}(p), U)$. Then $f(g^{-1}(p)) \in U$ and since $f(g^{-1}(p)) \notin f(V_2)$, $U \notin f(V_2)$. Hence $U \subseteq f(V_1)$ and $f^{-1}(U) \subseteq V_1 \subseteq V$. Consequently, $f^{-1}(p) \in V$.

**Proposition 4.** Let $(X, t)$ be a topological space and let $F$ be a collection of quasi-equicontinuous functions from $(X, t)$ into a quasi-uniform space $(Y, \mathcal{U})$. Then the topology of pointwise convergence on $F$ is jointly continuous.

**Proof.** Let $f \in F$ and let $x \in X$. For any $U \in \mathcal{U}$, $U(f(x))$ is a neighborhood of $f(x)$. Let $V \in \mathcal{U}$ such that $V \circ V \subseteq U$. By hypothesis there exists a neighborhood $N$ of $x$ such that for all $f \in F$, $f(N) \subseteq V(f(x))$. Consider the neighborhoods $W(x, V)(f)$ and $N$ of $f$ and $x$ respectively. Let $z \in N$ and let $g \in W(x, V)(f)$. Then $(f(x), g(x)), (g(x), g(z)) \in V$ and $g(z) \in V \circ V(f(x)) \subseteq U(f(x))$.

**Theorem 5** [2, Theorem 5]. Let $F$ be a semigroup (under composition) of continuous functions from a topological space $X$ into itself. If the topology of pointwise convergence on $F$ is jointly continuous, then composition is continuous relative to the topology of pointwise convergence.

**Theorem 6.** Let $(X, t)$ be any topological space and let $G$ be a group of homeomorphisms of $X$ onto $X$. Let $\mathcal{U}$ be any $R_3$-quasi-uniformity on $X$ such that $t \subseteq t_0$ and $G$ is quasi-equicontinuous with respect to $\mathcal{U}$. Then $G$ is a topological group under the topology of pointwise convergence.

**Proof.** By Proposition 4, the topology of pointwise convergence on $G$ is jointly continuous. Thus by Theorems 3 and 5, $G$ is a topological group under the topology of pointwise convergence.

We conclude by giving an example of a non-$R_0$ topological space $(X, t)$ with an $R_3$-quasi-uniformity $\mathcal{U}$ on $X$ such that $t \subseteq t_0$.

**Definition** [4]. A preorder on a set $X$ is any reflexive and transitive relation on $X$.

**Example.** Let $N$ denote the set of natural numbers. Let $\leq$ be an anti-symmetric preordering on $N$ defined as follows:
(i) \( x \leq x \) for all \( x \in N \),
(ii) \( 2 \leq 2^k, \ k = 1, 2, 3, \ldots \), and
(iii) \( 3 \leq 3^k, \ k = 1, 2, 3, \ldots \).

Let \( \tau \) be the left topology associated with the preordering \( \leq \) [4]. It is not difficult to see that \((N, \tau)\) is a \( T_0 \) space which is not \( T_1 \) and hence not \( R_0 \) [5, Corollary 3.9]. Let \( U_n = \{(x, y)|x = y \text{ or } x \geq n\} \), \( \beta = \{U_n|n \in N\} \) and \( \mathcal{V} \) denote the quasi-uniformity on \( N \) generated by the base \( \beta \) [1]. Then \( \mathcal{V} \) is an \( R_3 \)-quasi-uniformity on \( N \) with the property that \( \tau \) is properly contained in \( \tau_{\mathcal{V}} \).

REFERENCES


DEPARTMENT OF MATHEMATICS, NATIONAL UNIVERSITY OF IRAN, EVEEN, TEHERAN, IRAN