PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 52, October 1975

R_3-QUASI-UNIFORM SPACES AND TOPOLOGICAL HOMEOMORPHISM GROUPS[1]

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ABSTRACT. It is well known that if X is a completely regular space and G is a homeomorphism group of X onto itself such that G is equicontinuous with respect to a compatible uniformity of X, then G is a topological group under the topology of pointwise convergence. In this paper, we obtain a generalization of the above result by means of R_3-quasi-uniformities.

1. **Introduction.** Let (X, τ) be a topological space. It is well known that if $\mathfrak{U}$ is a compatible uniformity on X such that G is a homeomorphism group that is equicontinuous with respect to $\mathfrak{U}$, then G is a topological group under the topology of pointwise convergence. R. V. Fuller has obtained an analogous result for regular spaces [2] and we have shown previously that a similar result applies when (X, τ) is only an R_0 space (and hence, in particular, if X is T_1 or regular) [6]. In this paper we use R_3-quasi-uniformities to complement Fuller's result. We take the domain space (X, τ) to be an arbitrary topological space. If our domain space is regular, it is known that there exists a compatible R_3-quasi-uniformity $\mathfrak{U}$ on X, that is, $\tau = \tau_{\mathfrak{U}}$ [5]. Finally we give a simple example of a non-R_0 topological space (hence not regular) for which our principal result, Theorem 6, obtains.

Let Y be a topological space. A collection $\mathfrak{V}^*$ of two-element open covers of Y is said to be a *semiuniformity* for Y if for each $q \in Y$ and each neighborhood V of q there is $\{V_1, V_2\}$ in $\mathfrak{V}^*$ such that $q \in V_1 \subset V$ and $Y - V_2$ is a neighborhood of q [2]. Let F be a family of functions from a topological space X to semiuniform space $(Y, \mathfrak{V}^*)$. Then F is *semiequicontinuous* if for each $V \in \mathfrak{V}^*$ there is an open cover $\mathfrak{A}$ of X such that $\mathfrak{A}$ refines $f^{-1}(V)$ for each $f \in F$ [2]. One may easily show that a topological space has a semiuniformity if and only if it is regular.

Let X be a nonempty set. A *quasi-uniformity* for X is a filter $\mathfrak{U}$ of reflexive subsets of $X \times X$ such that if $U \in \mathfrak{U}$, there is $V \in \mathfrak{U}$ such that $V \circ V \subset U$ [5]. Let G be a collection of maps from a topological space (X, τ) into a quasi-uniform space $(Y, \mathfrak{U})$ and let $x \in X$. Then F *is quasi-equicontinuous at* x provided that for each $U \in \mathfrak{U}$ there exists a neighbor-

Received by the editors March 13, 1973 and, in revised form, December 1, 1973.
AMS (MOS) subject classifications (1970). Primary 54E15; Secondary 54C35.
Key words and phrases. R_3-quasi-uniform space, quasi-equicontinuous, quasi-uniform cover, joint continuity.
[1]This paper is based upon the author's dissertation at Virginia Polytechnic Institute and State University, Blacksburg, Virginia, under the direction of P. Fletcher.

hood N of x such that for $f \in F$, $f(N) \subset U(f(x))$ and F is quasi-equicontinuous provided F is quasi-equicontinuous at each $x \in X$. If $y \in Y$ and $U_1 \in \mathfrak{U}$ such that $U_1(y)$ is open and $U_2 \in \mathfrak{U}$ such that $U_2 \circ U_2 \circ U_2 \circ U_2(y) \subset U_1(y)$ and $U_2 = U_2^{-1}$, then $l = \{U_1(y), \bigcup\{\text{int } U_2(p): p \notin U_2 \circ U_2(y)\}\}$ is a *two element quasi-uniform cover of* X. A quasi-uniform space $(X, \mathfrak{U})$ is R_3, if, given $x \in X$ and $U \in \mathfrak{U}$, there exists a symmetric $W \in \mathfrak{U}$ such that $W \circ W(x) \subset U(x)$ [3]. It is shown that if (X, τ) is regular, then the Pervin quasi-uniformity on X is R_3 [5, Theorem 3.17].

2. Topological groups of homeomorphisms.

Theorem 1. *Let $(Y, \mathfrak{U})$ be an R_3-quasi-uniform space. Then the collection of all two element quasi-uniform covers of Y is a semiuniformity for Y.*

Proof. Let $q \in Y$ and let V be a neighborhood of q. Let $U_1 \in \mathfrak{U}$ such that $U_1(q) \subset V$ and $U_1(q)$ is open. By hypothesis there is a symmetric entourage $U_2 \in \mathfrak{U}$ such that $U_2 \circ U_2 \circ U_2 \circ U_2(q) \subset U_1(q)$. Let $\mathcal{C} = \{U_1(q), \bigcup\{\text{int } U_2(y): y \notin U_2 \circ U_2(q)\}\}$. Suppose that $x \in Y$ and $x \notin U_1(q)$. Note that if $z \in Y$ and $z \in U_2 \circ U_2(q)$, then $U_2(z) \subset U_1(q)$. Thus $x \notin U_2 \circ U_2(q)$ and $x \in \text{int } U_2(x)$. Therefore $\mathcal{C}$ is an open cover of Y. Furthermore, let $p \in U_2(q)$ and suppose that $p \in V_2 = \bigcup\{\text{int } U_2(y): y \notin U_2 \circ U_2(q)\}$. Then there exists a $y \in Y$ such that $p \in \text{int } U_2(y)$ and $y \notin U_2 \circ U_2(q)$. But $y \in U_2(p) \subset U_2 \circ U_2(q)$—a contradiction. Then $\mathcal{V}^* = \{\mathcal{C}: q \in Y$ and V is a neighborhood of $q\}$ is a semiuniformity for Y.

The semiuniformity $\mathcal{V}^*$ of the preceding theorem will be called a *quasi-uniform semiuniformity.*

Theorem 2. *Let $(Y, \mathcal{V})$ be an R_3-quasi-uniform space and let F be a family of quasi-equicontinuous functions from a topological space (X, τ) into $(Y, \mathcal{V})$. Then F is semiequicontinuous with respect to the quasi-uniform semiuniformity of $\mathcal{V}$.*

Proof. Let $\mathcal{V}^*$ be the quasi-uniform semiuniformity of $\mathcal{V}$, let $y, q \in Y$ and $U_1, U_2 \in \mathcal{V}$. Let $l \in \mathcal{V}^*$ such that $l = \{U_1(q), \bigcup\{\text{int } U_2(y): y \notin U_2 \circ U_2(q)\}\}$. By hypothesis, for each $x \in X$ there exists a neighborhood N_x of x such that for all $f \in F$, $f(N_x) \subset U_2(f(x))$. It may be seen that $U_2(f(x))$ is contained in either $U_1(q)$ or $V_2 = \bigcup\{\text{int } U_2(y): y \notin U_2 \circ U_2(q)\}$ as follows: Let $z_1, z_2 \in U_2(f(x))$, so that $z_1 \notin U_1(q)$ and $z_2 \notin V_2$. Now if $z_2 \notin V_2$, then $z_2 \in U_2 \circ U_2(q)$ and $(q, z_2) \in U_2 \circ U_2$. Since $(z_2, f(x)) \in U_2$ and $(f(x), z_1) \in U_2$, $z_1 \in U_2 \circ U_2 \circ U_2 \circ U_2(q) \subset U_1(q)$—a contradiction. Thus $\{N_x: x \in X\}$ is the desired open cover of X.

The proof of the following theorem is based on the proof of [2, Theorem 4].

Theorem 3. *Let F be a family of one-to-one functions of a topological space (X, τ) onto itself. Let $\mathcal{V}$ be an R_3-quasi-uniformity on X such that*

$\tau \subseteq \tau_{\mathcal{V}}$, *where* $\tau_{\mathcal{V}}$ *is the topology induced by* $\mathcal{V}$. *If* F^{-1} *is* $\mathcal{V}$-*quasi-equicontinuous, then the mapping* $\Psi: F \to F$, *defined by* $\Psi(f) = f^{-1}$, *is continuous relative to the topology of pointwise convergence on* F *and* F^{-1}.

Proof. Throughout the proof, if $p \in X$ and $U \in \tau$, then $W(p, U)$ denotes $\{f \in F: f(p) \in U\}$. Let $\mathcal{V}^*$ be the quasi-uniform semiuniformity of $\mathcal{V}$. Let $g \in F$, $p \in X$ and $V \in \tau$ such that $W(p, V)$ is a neighborhood of g^{-1}. Since $\tau \subseteq \tau_{\mathcal{V}}$ there is $\{V_1, V_2\} \in \mathcal{V}^*$ such that $g^{-1}(p) \subset V_1 \subset V$ and $X - V_2$ is a $\tau_{\mathcal{V}}$ neighborhood of $g^{-1}(p)$. By Theorem 2, F^{-1} is semiequicontinuous with respect to $\mathcal{V}^*$. Let $\mathcal{U}$ be a τ-open cover of X such that $\mathcal{U}$ refines $\{f(V_1), f(V_2)\}$ for all $f \in F$ and let U be a member of $\mathcal{U}$ that contains p. Then $W(g^{-1}(p), U)$ is a neighborhood of g. Let $f \in F$ such that $f \in W(g^{-1}(p), U)$. Then $f(g^{-1}(p)) \in U$ and since $f(g^{-1}(p)) \notin f(V_2)$, $U \not\subset f(V_2)$. Hence $U \subset f(V_1)$ and $f^{-1}(U) \subset V_1 \subset V$. Consequently, $f^{-1}(p) \in V$.

Proposition 4. *Let* (X, τ) *be a topological space and let* F *be a collection of quasi-equicontinuous functions from* (X, τ) *into a quasi-uniform space* $(Y, \mathcal{U})$. *Then the topology of pointwise convergence on* F *is jointly continuous.*

Proof. Let $f \in F$ and let $x \in X$. For any $U \in \mathcal{U}$, $U(f(x))$ is a neighborhood of $f(x)$. Let $V \in \mathcal{U}$ such that $V \circ V \subset U$. By hypothesis there exists a neighborhood N of x such that for all $f \in F$, $f(N) \subset V(f(x))$. Consider the neighborhoods $W(x, V)(f)$ and N of f and x respectively. Let $z \in N$ and let $g \in W(x, V)(f)$. Then $(f(x), g(x)), (g(x), g(z)) \in V$ and $g(z) \in V \circ V(f(x)) \subset U(f(x))$.

Theorem 5 [2, Theorem 5]. *Let* F *be a semigroup* (*under composition*) *of continuous functions from a topological space* X *into itself. If the topology of pointwise convergence on* F *is jointly continuous, then composition is continuous relative to the topology of pointwise convergence.*

Theorem 6. *Let* (X, τ) *be any topological space and let* G *be a group of homeomorphisms of* X *onto* X. *Let* $\mathcal{V}$ *be any* R_3-*quasi-uniformity on* X *such that* $\tau \subseteq \tau_{\mathcal{V}}$ *and* G *is quasi-equicontinuous with respect to* $\mathcal{V}$. *Then* G *is a topological group under the topology of pointwise convergence.*

Proof. By Proposition 4, the topology of pointwise convergence on G is jointly continuous. Thus by Theorems 3 and 5, G is a topological group under the topology of pointwise convergence.

We conclude by giving an example of a non-R_0 topological space (X, τ) with an R_3-quasi-uniformity $\mathcal{V}$ on X such that $\tau \subset \tau_{\mathcal{V}}$.

Definition [4]. A *preorder* on a set X is any reflexive and transitive relation on X.

Example. Let N denote the set of natural numbers. Let $\leq$ be an antisymmetric preordering on N defined as follows:

(i) $x \leq x$ for all $x \in N$,

(ii) $2 \leq 2^k$, $k = 1, 2, 3, \ldots$, and

(iii) $3 \leq 3^k$, $k = 1, 2, 3, \ldots$.

Let τ be the left topology associated with the preordering $\leq$ [4]. It is not difficult to see that (N, τ) is a T_0 space which is not T_1 and hence not R_0 [5, Corollary 3.9]. Let $U_n = \{(x, y) | x = y \text{ or } x \geq n\}$, $\beta = \{U_n | n \in N\}$ and $\mathcal{U}$ denote the quasi-uniformity on N generated by the base β [1]. Then $\mathcal{U}$ is an R_3-quasi-uniformity on N with the property that τ is properly contained in $\tau_{\mathcal{U}}$.

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