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NONSTRATIFIABLE REGULAR QUOTIENTS OF SEPARABLE STRATIFIABLE SPACES

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ABSTRACT. A regular quotient of a countable M_1-space need not be stratifiable, and a regular first countable quotient of a first countable separable M_1-space need not be stratifiable either. This answers a question of Borges. The second example depends on a new, simple, example of a first countable separable nonmetrizable M_1-space.

1. **Introduction.** Borges [B_1], [B_2] asked whether the regular quotient of a separable stratifiable space must be stratifiable. This question is related to the fact that a first countable regular quotient of a separable metrizable space is metrizable, but that the regular quotient of a nonseparable metrizable space need not be stratifiable, [S, corollary on p. 695, and §6]. We give two examples which settle this question negatively: neither the regular quotient of a countable stratifiable space nor the first countable (separable) quotient of a first countable separable stratifiable space need be stratifiable. In fact, both quotient spaces are not even monotonically normal, and both domains are hereditarily an M_1-space. [Recall that it is not known whether a stratifiable space must be M_1, nor whether a closed subspace of an M_1-space must be an M_1-space.]

Our examples leave open Michael's question, recorded in [B_2], whether the regular quotient of a separable metrizable space must be stratifiable. After seeing the first example, Michael asked whether a regular countable k-space must be stratifiable or an $\aleph_0$-space. [Recall that regular quotients of separable metrizable spaces are precisely the spaces which are both k-space and $\aleph_0$-space [M, Corollary 11.5 on p. 999]. Note that a countable k-space is sequential, hence such a space is the quotient of a (not necessarily separable) metrizable space [F, 1.12 on p. 112].] Of these questions, the first remains open, the second will be settled negatively in [vD_3].

All spaces in this note are T_1-spaces. We assume that the reader is familiar with the concepts used above, adequate references are [B_1], [C], [HLZ], [M] and [S]. R denotes the real numbers.

2. **The examples.** Heath [H_2] has given an example of a countable regular nonstratifiable space, the author has given another example in [vD_2] (it

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is shown in [vD$_3$] that these examples really are different).

2.1. **Example.** *A regular quotient of a countable M_1-space which is not stratifiable.*

By virtue of the preceding remarks, it suffices to prove the following

2.2. **Proposition.** *Every countable space is the quotient of a countable M_1-space.*

Proof. Let T be a countable space. For each $t \in T$, let T_t be the space with the same underlying set as T, which has

$$\mathcal{B}_t = \{U \subset T \mid U \text{ open and } t \in U\} \cup \{\{b\} \mid b \in T \setminus \{t\}\}$$

as base. Since T is a T_1-space, T_t is a regular T_1-space. Hence T_t is an M_1-space, since $\mathcal{B}_t$ is σ-closure preserving.

Let T_∞ be the topological sum of the family $\{T_t \times \{t\} \mid t \in T\}$, so that $T \times T$ is the underlying set of T_∞. Define a map $\pi\colon T_\infty \to T$ by $\pi(x, y) = x$. Then π is a quotient map. [In fact, π is even hereditarily quotient, i.e. $\pi | \pi^{-1}[S]$ is a quotient map from $\pi^{-1}[S]$ onto S for each $S \subset T$.]

The quotient space in Example 2.1 is not monotonically normal, for a countable space is stratifiable if it is monotonically normal, cf. [HLZ, Example 7.3 on p. 490].

The second example is based on a similar idea. We first give an example of a first countable separable nonmetrizable M_1-space which is much simpler than Ceder's example [C, Example 9.2, p. 122].

2.3. **Example.** *A first countable separable M_1-space which is not metrizable.*

Description. The underlying set of our example is the subset $M = X \cup Q$ of R^2 (the plane), where

$X = R \times \{0\}$ (the x-axis),

$Q = \{\langle x, y\rangle \mid \text{both } x \text{ and } y \text{ are rational}, y > 0\}$.

The topology of M is determined as follows. Points of Q are isolated, and a basic neighborhood of a point $p = \langle x, 0\rangle \in X$ is of the form

$$B_{a,b}(p) = \{\langle s, t\rangle \in M \mid a < s < b,\ t \le |x - s|\}$$

where $a < x < b$. Clearly M is regular. For $a, b \in R$ with $a < b$ the family

$$\mathcal{B}_{a,b} = \{B_{a,b}(p) \mid p = \langle x, 0\rangle \text{ with } a < x < b\}$$

is closure preserving, hence

$$\mathcal{B} = \bigcup \{\mathcal{B}_{a,b} \mid a, b \text{ rational and } a < b\} \cup \{\{q\} \mid q \in Q\}$$

is a σ-closure preserving base for M. Therefore M is an M_1-space. Similarly each subspace is an M_1-space. M is first countable and separable, but is not second countable. Hence M is not metrizable.

2.4. **Example.** *A regular first countable separable quotient space of a first countable separable M_1-space, which is not monotonically normal.*

Description. We will consider several topologies on the subset $P \cup Q$ of $R \times R$, where

$P = \{\langle x, 0\rangle | x$ irrational$\}$,

$Q = \{\langle x, y\rangle |$both x and y are rational, $y > 0\}$.

The associated spaces will be called A and, for each $t \in Q$, B_t. In all these spaces, a basic neighborhood of a point $p = \langle x, 0\rangle \in X$ is of the form

$$B(p, \epsilon) = \{\langle s, t\rangle \in P \cup Q | t \leq |x - s| < \epsilon\}$$

where $\epsilon > 0$. In A points of Q have their Euclidean neighborhoods, in B_t all points of $Q\backslash\{t\}$ are isolated, while t has its Euclidean neighborhoods. Then A and B_t are first countable cosmic (see [M, §10]) spaces.

A was introduced by Heath [H_1] as an example of a semimetrizable paracompact space which is not stratifiable. Since cosmic spaces are semistratifiable and a space is stratifiable if and only if it is semistratifiable and monotonically normal [HLZ, Theorem 2.5], A is not monotonically normal (there is also a direct proof by a category argument, via condition (b) of [HLZ, Lemma 2.2]).

Observing that $B_t\backslash\{t\}$ is a subspace of M, we easily see that (each subspace of) B_t is an M_1-space. In a similar way as in Proposition 2.2, we can define a (hereditarily) quotient map from the topological sum B_∞ of the family $\{B_t | t \in Q\}$ onto A. B_∞ is a first countable separable M_1-space.

2.5. **Remarks.** (a) The fact that A is cosmic is not accidental: by a theorem of Michael, every separable stratifiable space is cosmic, see [B_2, 2.2.A], hence the regular quotient of such a space must be cosmic, too.

(b) One can show in almost the same way that example H_1 of [vD_1, 4.1] is the regular quotient of a first countable separable M_1-space.

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