

A HANDLEBODY WITH ONE PILLBOX HAS NO FAKE 3-CELLS

J. O. BERGE AND MATTHEW G. BRIN

ABSTRACT. The main result establishes upper bounds on the rank of H_1 of orientable 3-manifolds with fake 3-cells and on the rank of π_1 of closed orientable 3-manifolds with fake 3-cells. As a consequence we get that a 3-manifold obtained by sewing one pillbox on a handlebody of arbitrary genus cannot contain a fake 3-cell.

1. Introduction. The Poincaré conjecture, an open question of long standing in the topology of 3-manifolds, states that any compact 3-manifold without boundary that is simply connected must be a 3-sphere. An equivalent statement is that a compact, simply connected 3-manifold with connected, 2-sphere boundary is a 3-cell. A counterexample to the second statement is called a *fake 3-cell*. Since any orientable 3-manifold may be obtained by sewing pillboxes onto a handlebody of sufficient genus, it is of interest to know which objects of this type cannot contain a fake 3-cell. C. D. Feustel has shown in [1] that if a handlebody with one pillbox has free fundamental group, then the resulting manifold is also a handlebody, and thus does not contain a fake 3-cell. We show that sewing a single pillbox onto a handlebody in any fashion gives a manifold that contains no fake 3-cells. This is obtained as a corollary to Theorem 3 below. This theorem is based on the fact that lens spaces contain no fake 3-cells, and on a result of Haken that allows us to induct from there. A restatement of a weakened form of Haken's theorem [3, Theorem, p. 84] is as follows:

Let a closed orientable 3-manifold M be the union of two handlebodies H_1 and H_2 whose intersection T is their common boundary. Suppose M contains a 2-sphere that bounds no 3-cell. Then M contains a 2-sphere S that bounds no 3-cell and so that $S \cap T$ is a simple closed curve.

In the following we will work in the category of simplicial complexes and peicewise linear maps. All manifolds are compact and orientable, all homologies will be over Z , the rank of homology groups will be the free rank and the rank of homotopy groups will be the minimal number of generators needed for a presentation. A closed manifold will be compact and without boundary. By a Heegaard splitting of genus n we will mean two 3-dimensional handlebodies of genus n sewn together along their boundaries by a

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homeomorphism. All closed, orientable 3-manifolds can be represented as a Heegaard splitting of genus n for some n . If X is a closed, orientable 3-manifold, then the *Heegaard genus* of X is the smallest integer n so that X admits a Heegaard splitting of genus n . A *pillbox* will be a 3-cell regarded as the cross product of a 2-cell, D^2 , and the unit interval, I . A 3-manifold X will be called a *handlebody with pillboxes* if it is obtained by sewing a finite number of pillboxes onto a handlebody H by homeomorphisms identifying the annuli $\partial D^2 \times I$ with disjoint annuli contained in ∂H . The genus of such an X will be the genus of H . If X is a handlebody with pillboxes, of genus n , or a closed 3-manifold of Heegaard genus n , then the *homology deficiency* of X will be $n - \text{rank}[H_1(X)]$; the *homotopy deficiency* will be $n - \text{rank}[\pi_1(X)]$. Note that both numbers are nonnegative and that $\text{rank} \pi_1(X) \geq \text{rank} H_1(X)$.

2. **Lemma 1.** *Let X be a compact, orientable 3-manifold with connected boundary of genus g . Then*

(a) $\text{rank} H_1(X) \geq g$,

(b) *there is a handlebody H of genus g and a homeomorphism $h: \partial X \rightarrow \partial H$ so that $\text{rank} H_1(X \cup_h H) = \text{rank} H_1(X)$.*

Proof. For orientable manifolds we know, for homology over Z , that $2g = \text{rank} H_1(\partial X) =$ twice the rank of $[\text{Image } i_*: H_1(\partial X) \rightarrow H_1(X)]$ so (a) is true. Let j_* be the projection from $[\text{Image } i_*]$ onto its free part. j_*i_* is a map from a free abelian group of rank $2g$ onto a free abelian group of rank g and we have $H_1(\partial X) \cong \text{Ker}(j_*i_*) \oplus \text{Im}(j_*i_*)$.

Let α be a primitive element in $\text{Ker}(j_*i_*)$. α can be represented by a simple closed curve, J , on ∂X that does not separate ∂X [5, Proposition 2.11]. If we obtain a new manifold \tilde{X} by sewing a pillbox onto X along a regular neighborhood of J in ∂X , we will kill $i_*(\alpha)$. However $i_*(\alpha)$ was either zero or torsion so that $\text{rank} H_1(\tilde{X}) = \text{rank} H_1(X)$. Since $\partial \tilde{X}$ is connected and of smaller genus than ∂X we have (b) by induction.

Corollary 2. *Let X be a handlebody H with pillboxes such that ∂X is connected. Then X can be embedded in a closed 3-manifold M with Heegaard splitting $(H, \overline{M-H})$ such that *homology deficiency* of $M =$ *homology deficiency* of X .*

Theorem 3. *Let X be a handlebody H with pillboxes or a closed 3-manifold with Heegaard splitting $(H, \overline{X-H})$. If X contains a fake 3-cell, then the *homology deficiency* of X is at least 2.*

Proof.

Case 1. X is closed or $\partial X = S^2$. We can assume that X is closed since we can always sew a 3-cell onto ∂X . Let C be a fake 3-cell in X and let $D = \overline{X-C}$. Since lens spaces contain no fake cells we can assume that $\text{genus } X \geq 2$. If X is a homotopy 3-sphere, $\text{genus } X \geq 2$ and the desired

result follows; hence we may suppose that

$$1 \neq \pi_1(X) = \pi_1(C) * \pi_1(D) = \pi_1(D).$$

Thus ∂C is a 2-sphere that does not bound a 3-cell. Since genus $X \geq 2$ we can use Haken's theorem to find a 2-sphere S in X so that $S \cap H$ and $S \cap (\overline{X-H})$ are properly embedded disks separating each of H and $\overline{X-H}$ into two handlebodies of nonzero genus. If genus $X = 2$ this gives a connected sum of lens spaces which contradicts the hypothesis, so $\pi_1(X) = 1$. If genus $X > 2$ let the complementary domains of S be M_1 and M_2 . It is a consequence of Milnor's prime decomposition theorem [4] that either M_1 or M_2 contains a fake 3-cell and by induction has homology deficiency at least 2 (see also [2, Theorem 2]). $H_1(X) = H_1(M_1) \oplus H_1(M_2)$ and the theorem follows.

Case 2. ∂X is connected. Follows from Corollary 2 and Case 1.

Case 3. ∂X is not connected. We will connect up ∂X by adding 1-handles to H along pairs of disks disjoint from the annuli that ∂H shares with the pillboxes. Each 1-handle will be sewn to $\partial X \cap \partial H$ along a pair of disks in different components of ∂X , and enough 1-handles will be added so that the resulting manifold has connected boundary. Since each handle increases genus H by one and rank $H_1(X)$ by one, the deficiency is the same and we are reduced to Case 2. This completes the proof.

The proper substitutions in the proof of Case 1, replacing homology by homotopy, H_1 by π_1 , and direct product by free product, will give:

Theorem 4. *Let a closed orientable 3-manifold M have a fake 3-cell. Then the homotopy deficiency of M is ≥ 2 .*

Corollary 5. *Let X be a handlebody plus one pillbox. Then X contains no fake 3-cells.*

Proof. Lemma 1(a) and Theorem 3.

Note. If m is the smallest integer such that a homotopy 3-sphere exists of Heegaard genus m , then Theorems 3 and 4 can be restated with deficiency m , and Corollary 5 can be restated with $(m - 1)$ pillboxes.

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DEPARTMENT OF MATHEMATICS, UNIVERSITY OF WISCONSIN, MADISON, WISCONSIN