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A HANDLEBODY WITH ONE PILLBOX HAS NO FAKE 3-CELLS

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ABSTRACT. The main result establishes upper bounds on the rank of H_1 of orientable 3-manifolds with fake 3-cells and on the rank of π_1 of closed orientable 3-manifolds with fake 3-cells. As a consequence we get that a 3-manifold obtained by sewing one pillbox on a handlebody of arbitrary genus cannot contain a fake 3-cell.

1. **Introduction.** The Poincaré conjecture, an open question of long standing in the topology of 3-manifolds, states that any compact 3-manifold without boundary that is simply connected must be a 3-sphere. An equivalent statement is that a compact, simply connected 3-manifold with connected, 2-sphere boundary is a 3-cell. A counterexample to the second statement is called a *fake 3-cell*. Since any orientable 3-manifold may be obtained by sewing pillboxes onto a handlebody of sufficient genus, it is of interest to know which objects of this type cannot contain a fake 3-cell. C. D. Feustel has shown in [1] that if a handlebody with one pillbox has free fundamental group, then the resulting manifold is also a handlebody, and thus does not contain a fake 3-cell. We show that sewing a single pillbox onto a handlebody in any fashion gives a manifold that contains no fake 3-cells. This is obtained as a corollary to Theorem 3 below. This theorem is based on the fact that lens spaces contain no fake 3-cells, and on a result of Haken that allows us to induct from there. A restatement of a weakened form of Haken's theorem [3, Theorem, p. 84] is as follows:

Let a closed orientable 3-manifold M be the union of two handlebodies H_1 and H_2 whose intersection T is their common boundary. Suppose M contains a 2-sphere that bounds no 3-cell. Then M contains a 2-sphere S that bounds no 3-cell and so that $S \cap T$ is a simple closed curve.

In the following we will work in the category of simplicial complexes and peicewise linear maps. All manifolds are compact and orientable, all homologies will be over Z, the rank of homology groups will be the free rank and the rank of homotopy groups will be the minimal number of generators needed for a presentation. A closed manifold will be compact and without boundary. By a Heegaard splitting of genus n we will mean two 3-dimensional handlebodies of genus n sewn together along their boundaries by a

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homeomorphism. All closed, orientable 3-manifolds can be represented as a Heegaard splitting of genus n for some n. If X is a closed, orientable 3-manifold, then the *Heegaard genus* of X is the smallest integer n so that X admits a Heegaard splitting of genus n. A *pillbox* will be a 3-cell regarded as the cross product of a 2-cell, D^2, and the unit interval, I. A 3-manifold X will be called a *handlebody with pillboxes* if it is obtained be sewing a finite number of pillboxes onto a handlebody H by homeomorphisms identifying the annuli $\partial D^2 \times I$ with disjoint annuli contained in ∂H. The genus of such an X will be the genus of H. If X is a handlebody with pillboxes, of genus n, or a closed 3-manifold of Heegaard genus n, then the *homology deficiency* of X will be $n - \operatorname{rank}[H_1(X)]$; the *homotopy deficiency* will be $n - \operatorname{rank}[\pi_1(X)]$. Note that both numbers are nonnegative and that $\operatorname{rank} \pi_1(X) \geq \operatorname{rank} H_1(X)$.

2. **Lemma 1.** *Let X be a compact, orientable 3-manifold with connected boundary of genus g. Then*

(a) rank $H_1(X) \geq g$,

(b) *there is a handlebody H of genus g and a homeomorphism $h: \partial X \longrightarrow \partial H$ so that* rank $H_1(X \cup_h H) = \operatorname{rank} H_1(X)$.

Proof. For orientable manifolds we know, for homology over Z, that $2g = \operatorname{rank} H_1(\partial X) =$ twice the rank of [Image $i_*: H_1(\partial X) \longrightarrow H_1(X)$] so (a) is true. Let j_* be the projection from [Image i_*] onto its free part. $j_* i_*$ is a map from a free abelian group of rank $2g$ onto a free abelian group of rank g and we have $H_1(\partial X) \cong \operatorname{Ker}(j_* i_*) \oplus \operatorname{Im}(j_* i_*)$.

Let α be a primitive element in $\operatorname{Ker}(j_* i_*)$. α can be represented by a simple closed curve, J, on ∂X that does not separate ∂X [5, Proposition 2.11]. If we obtain a new manifold $\tilde{X}$ by sewing a pillbox onto X along a regular neighborhood of J in ∂X, we will kill $i_*(\alpha)$. However $i_*(\alpha)$ was either zero or torsion so that rank $H_1(\tilde{X}) = \operatorname{rank} H_1(X)$. Since $\partial\tilde{X}$ is connected and of smaller genus than ∂X we have (b) by induction.

Corollary 2. *Let X be a handlebody H with pillboxes such that ∂X is connected. Then X can be embedded in a closed 3-manifold M with Heegaard splitting $(H, \overline{M - H})$ such that homology deficiency of M = homology deficiency of X.*

Theorem 3. *Let X be a handlebody H with pillboxes or a closed 3-manifold with Heegaard splitting $(H, \overline{X - H})$. If X contains a fake 3-cell, then the homology deficiency of X is at least 2.*

Proof.

Case 1. *X is closed or $\partial X = S^2$.* We can assume that X is closed since we can always sew a 3-cell onto ∂X. Let C be a fake 3-cell in X and let $D = \overline{X - C}$. Since lens spaces contain no fake cells we can assume that genus $X \geq 2$. If X is a homotopy 3-sphere, genus $X \geq 2$ and the desired result follows; hence we may suppose that

$$1 \neq \pi_1(X) = \pi_1(C) * \pi_1(D) = \pi_1(D).$$

Thus ∂C is a 2-sphere that does not bound a 3-cell. Since genus $X \geq 2$ we can use Haken's theorem to find a 2-sphere S in X so that $S \cap H$ and $S \cap \overline{(X - H)}$ are properly embedded disks separating each of H and $\overline{X - H}$ into two handlebodies of nonzero genus. If genus $X = 2$ this gives a connected sum of lens spaces which contradicts the hypothesis, so $\pi_1(X) = 1$. If genus $X > 2$ let the complementary domains of S be M_1 and M_2. It is a consequence of Milnor's prime decomposition theorem [4] that either M_1 or M_2 contains a fake 3-cell and by induction has homology deficiency at least 2 (see also [2, Theorem 2]). $H_1(X) = H_1(M_1) \oplus H_1(M_2)$ and the theorem follows.

Case 2. *∂X is connected.* Follows from Corollary 2 and Case 1.

Case 3. *∂X is not connected.* We will connect up ∂X by adding 1-handles to H along pairs of disks disjoint from the annuli that ∂H shares with the pillboxes. Each 1-handle will be sewn to $\partial X \cap \partial H$ along a pair of disks in different components of ∂X, and enough 1-handles will be added so that the resulting manifold has connected boundary. Since each handle increases genus H by one and rank $H_1(X)$ by one, the deficiency is the same and we are reduced to Case 2. This completes the proof.

The proper substitutions in the proof of Case 1, replacing homology by homotopy, H_1 by π_1, and direct product by free product, will give:

Theorem 4. *Let a closed orientable 3-manifold M have a fake 3-cell. Then the homotopy deficiency of M is ≥ 2.*

Corollary 5. *Let X be a handlebody plus one pillbox. Then X contains no fake 3-cells.*

Proof. Lemma 1(a) and Theorem 3.

Note. If m is the smallest integer such that a homotopy 3-sphere exists of Heegaard genus m, then Theorems 3 and 4 can be restated with deficiency m, and Corollary 5 can be restated with $(m - 1)$ pillboxes.

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