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ON THE EXISTENCE OF CONTACT FORMS

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ABSTRACT. Using an old theorem of Alexander, we give a short and elementary proof that every closed, orientable 3-manifold has a contact form.

Introduction. Let M^m $(m = 2n + 1)$ be a closed, smooth, orientable m-manifold; a contact form on M is a smooth 1-form ω such that $\omega \wedge (d\omega)^n \neq 0$ at each point. One knows [2] that the existence of a contact form on M^m allows a reduction of the structure group of the tangent bundle of M to $U(n)$ and hence, in particular, the odd-dimensional Stiefel-Whitney classes of M have to vanish [4]. This, however, gives no information in dimension 3 and in [3] Chern[1] asked: Does every closed, smooth, orientable 3-manifold admit a smooth contact form?

This question was first answered (in the affirmative) by Lutz [6] and Martinet [7]. They based their proof on the theorem that every closed, orientable 3-manifold can be obtained from S^3 by surgery, the same theorem used by Lickorish, Novikov and Zieschang to prove that every such manifold admits a codimension 1 foliation, i.e. a nonzero 1-form η on M such that $\eta \wedge d\eta \equiv 0$; that is, in dimension 3, the exact opposite of a contact form.

It was observed by one of us (see also [5]) that the existence theorem for codimension 1 foliations on M is an immediate consequence of the following

Theorem (Alexander [1]). *Every closed, smooth, orientable 3-manifold M^3 is diffeomorphic to $W(h) \cup_{\mathrm{id}} \partial W \times D^2$, where D^2 is the 2-disc and W is an orientable 2-manifold with boundary and $h\colon W \to W$ is a diffeomorphism which restricts to the identity on ∂W; $W(h)$ denotes the mapping torus of h, i.e. the 3-manifold with boundary obtained from $W \times I$ by identifying $(x, 0)$ with $(h(x), 1)$, $I = [0, 1]$.*

The object of this short note is to prove that the existence theorem for contact forms on 3-manifolds is also a more or less immediate and elementary consequence of Alexander's theorem.

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[1]Also in Proc. U. S.-Japan Seminar in Differential Geometry (Kyoto, 1965), Nippon Hyoronsha, Tokyo, 1966; Question 1, p. 172. MR 35 #7268.

Construction of a contact form. We may assume that h fixes a neighborhood of ∂W. Let $d\theta$ be a volume form for ∂W, and let t be the collar parameter for a collar neighborhood C of ∂W, in W, invariant by h.

Let α_1 be any 1-form which equals $(1+t)d\theta$ near ∂W. By Stokes' theorem, $\int_W d\alpha_1 = 1$. If Ω is any volume form for W which is $dt \wedge d\theta$ near ∂W, then, by de Rham's theorem, $\Omega - d\alpha_1 = d\beta$, where β is a 1-form which is zero near ∂W. Let $\alpha_2 = \alpha_1 + \beta$. Then α_2 satisfies (i) $d\alpha_2$ is a volume form, and (ii) $\alpha_2 = (1+t)d\theta$ near ∂W.

The set of 1-forms satisfying (i) and (ii) is convex. Hence, there is a 1-form α on $W(h)$, such that α restricted to any fiber, satisfies (i) and (ii). Since $d\theta$ and t are defined in a natural way on $C \times S^1$, we may also specify that $\alpha = (1+t)d\theta$ in a neighborhood of $\partial W(h)$.

For a sufficient large constant K, it follows that $\omega = \alpha + K d\phi$ is a contact form on $W(h)$, where $d\phi$ is the pullback of a volume form for the base of the bundle $W(h)$ over S^1. In fact, $d\phi \wedge d\alpha$ is a nonsingular 3-form on $W(h)$, so $\omega \wedge d\omega = K(d\phi \wedge d\alpha) + \alpha \wedge d\alpha$ will also be nonsingular when K is large.

We will now extend ω over $\partial W(h) \times D^2$. We will use coordinates (θ, r, ϕ), where (r, ϕ) are polar coordinates for D^2. We may consider $C \times S^1$ to be $\partial W(h) \times (D_2^2 - D^2)$, where D_2^2 is the disk of radius two, and the present coordinates are related to the old coordinates by $\phi = \phi$, $\theta = \theta$, $r = 1 + t$, so in the new coordinates, $\omega = r d\theta + K d\phi$ in $\partial W(h) \times (D_2^2 - D^2)$. The form $-d\theta + r^2 d\phi$ is a smooth contact form near $r = 0$: this may be checked by converting to cartesian coordinates for D^2. Consider a 1-form ω which may be written $\omega = f_1(r)d\theta + f_2(r)d\phi$. ω is a contact form away from $r = 0$, iff

$$\omega \wedge d\omega = \begin{vmatrix} f_1(r) & f_1'(r) \\ f_2(r) & f_2'(r) \end{vmatrix} d\theta \wedge dr \wedge d\phi \neq 0.$$

That is, ω is a contact form iff the position vector and the tangent vector of the curve $(f_1(r), f_2(r))$ in $\mathbf{R}^2$ are linearly independent, for $r \neq 0$. It is now clear that there is a curve $(f_1(r), f_2(r))$ $[0 < r \leq 2]$ such that $(f_1(r), f_2(r)) = (-1, r^2)$ near $r = 0$, $(f_1(r), f_2(r)) = (r, K)$ for $1 \leq r \leq 2$, and such that $\omega = f_1(r)d\theta + f_2(r)d\phi$ is a contact form.

This completes the construction.

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