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AN ELEMENTARY METHOD FOR ESTIMATING ERROR TERMS IN ADDITIVE NUMBER THEORY[1]

ELMER K. HAYASHI

ABSTRACT. Let $R_k(n)$ denote the number of ways of representing the integers not exceeding n as the sum of k members of a given sequence of nonnegative integers. Using only elementary methods, we prove a general theorem from which we deduce that, for every $\epsilon > 0$,

$$R_k(n) - cn^\beta \neq o(n^{\beta(1-\beta)(1-1/k)/(1-\beta+\beta/k)-\epsilon})$$

where c is a positive constant and $0 < \beta < 1$.

Let $R_k(n)$ denote the number of ways of representing the integers not exceeding n as the sum of k members of a given sequence of nonnegative integers. Jurkat [4] has shown that $R_k(n) - G(n) \neq o(n^{\beta/4})$ whenever k is an even integer, $0 < \beta < 2$, and $G(n)$ is a logarithmico-exponential function with $G(n) \sim cn^\beta$, $c > 0$. Randol [5] has shown that $R_k(n) - cn^\beta \neq o(n^{\beta(1-1/k)(1-\beta/k)})$ when $m = k/\beta$ is an even integer, the given sequence of nonnegative integers is the sequence $\{n^m\}_{n=1}^\infty$, and c is the volume of the k-dimensional solid defined by $y_1^m + y_2^m + \cdots + y_k^m \leq 1$. The corollary to our first theorem improves Jurkat's result in case $\beta < (3k-4)/(3k-3)$ and comes surprisingly close to Randol's result even though Randol's theorem deals only with a very special case of ours. In contrast to the methods employed by others on this type of problem (see [1]–[6]), the techniques we use here are all elementary.

We begin by defining our notation. Let $\{r_1(n)\}_{n=0}^\infty$ be a sequence of nonnegative real numbers such that if $r_1(n) \neq 0$, then $r_1(n) \geq 1$. (The lower bound 1 is chosen for convenience; any positive lower bound would suffice.) If k is an integer, $k \geq 2$, define $r_k(n)$ by

$$r_k(n) = \sum_{m_1+\cdots+m_k=n} r_1(m_1)\cdots r_1(m_k) = \sum_{m=0}^{n} r_1(m) r_{k-1}(n-m). \tag{1}$$

$R_k(n)$ will denote the summatory function of $r_k(n)$. Thus

$$R_k(n) = \sum_{m=0}^{n} r_k(m). \tag{2}$$

If $r_1(n)$ is a nonnegative integer for all n, we can interpret $r_1(n)$ as the number of occurrences of n in a given sequence $\{a_m\}$ of nonnegative integers.

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In this case, $r_k(n)$ denotes the number of ways n can be represented as the sum of k elements of the sequence $\{a_m\}$. It is interesting to note, however, that the proofs of our theorems do not require $r_1(n)$ to be integer-valued.

Finally, define $\Delta G(n) = G(n) - G(n-1)$ and $\Delta^2 G(n) = \Delta(\Delta G(n))$. We write $f(n) << g(n)$ when $f(n)$ is less than a positive multiple of $g(n)$ for all sufficiently large n. Our main result is

Theorem 1. *Let* $0 < \beta \le 1 - \delta < 1$. *If* $R_k(n) = G(n) + v(n)$, *with* $v(n) = o(G(n))$, *if* $n^\beta << G(n) << n^{1-\delta}$, *and if* $\Delta^2 G(n) \le 0$ *for all sufficiently large* n, *then for every* $\epsilon > 0$ *we have*

$$v(n) \ne o(n^{\delta\beta(1-1/k)/(1-\beta+\beta/k)-\epsilon}).$$

Proof. By hypothesis, there exists $n_0 > 0$ such that $\Delta^2 G(n) \le 0$ for $n \ge n_0$, i.e., $\Delta G(n)$ is nonincreasing for $n \ge n_0$. Thus for $n \ge n_0$ we have

$$(3) \qquad (n - n_0)\Delta G(n) \le \sum_{m=n_0+1}^{n} \Delta G(m) = G(n) - G(n_0),$$

and hence

$$(4) \qquad \Delta G(n) << G(n)/n.$$

Choose $x \in Z^+$ such that $r_1(x+1) \ne 0$ and assume $2 < y < x$, where y is an integer to be specified later. Since $R_k(n) \sim G(n)$ and $G(n) \to \infty$ as $n \to \infty$, there exist arbitrarily large n for which $r_1(n) \ne 0$. Hence x can be taken arbitrarily large. Using (2) and (1), we see that

$$R_k(x+y) - R_k(x) = \sum_{m=x+1}^{x+y} r_k(m) = \sum_{m=x+1}^{x+y} \sum_{j=0}^{m} r_1(j) r_{k-1}(m-j)$$

$$\ge r_1(x+1) \sum_{m=0}^{y-1} r_{k-1}(m) = r_1(x+1) R_{k-1}(y-1).$$

At first glance, the above estimate might seem to be rather crude. However, since by hypothesis $G(n) = o(n)$, and hence $R_k(n) = o(n)$, it follows that it is very unlikely that $r_1(n) \ne 0$ for very many n between x and $2x$. Thus the above estimate is good when $R_k(n)$ is significantly smaller than n in magnitude.

For the moment, let us assume that we know

$$(5) \qquad R_{k-1}(y) >> (R_k(y))^{1-1/k}.$$

Now using $G(n) >> n^\beta$ and recalling $R_k(n) \sim G(n)$, we obtain, for sufficiently large y,

$$R_k(x+y) - R_k(x) >> (R_k(y-1))^{1-1/k} >> (G(y-1))^{1-1/k} >> y^{\beta(1-1/k)}.$$

On the other hand, if $v(n) = o(n^\alpha)$, then using the fact that $\Delta G(n)$ is nonincreasing and applying (4), we get

$$R_k(x+y) - R_k(x) = G(x+y) - G(x) + v(x+y) - v(x)$$
$$\leq y\Delta G(x) + o(x^{\alpha}) << yG(x)/x + o(x^{\alpha}).$$

Thus we have, for sufficiently large y,

(6) $$y^{\beta(1-1/k)} << yG(x)/x + o(x^{\alpha}).$$

We now choose y so that

(7) $$yG(x)/x = o(y^{\beta(1-1/k)}),$$

say

$$y = [(x/G(x))^{1/(1-\beta+\beta/k)-\epsilon}], \qquad \epsilon > 0.$$

It is easily verified that $y < x$, and when ϵ is small, y grows large with x. With our choice of y, we see that a contradiction of (6) will occur if $\alpha < \delta\beta(1-1/k)/(1-\beta+\beta/k)$, since then for sufficiently small ϵ we have

$$x^{\alpha} \leq x^{\delta\beta(1-1/k)(1/(1-\beta+\beta/k)-\epsilon)}$$
$$<< (x/G(x))^{\beta(1-1/k)(1/(1-\beta+\beta/k)-\epsilon)} << y^{\beta(1-1/k)}.$$

The proof of the theorem will be complete if we can verify (5). In fact we shall show

(8) $$(R_k(y))^{k-1} \leq (k-1)(R_{k-1}(y))^k.$$

We write

$$(R_k(y))^{k-1} = \left(\sum_{a_1+\cdots+a_k = y} r_1(a_1)\cdots r_1(a_k)\right)^{k-1}$$

If we multiply out the right side of the last equation, we see that a typical term of $(R_k(y))^{k-1}$ is $\Pi_{i=1}^{k-1}\{\Pi_{j=1}^{k} r_1(a_{ij})\}$, where $\Sigma_{j=1}^{k} a_{ij} \leq y$ for $i = 1, 2, \cdots, k-1$. Now

$$\sum_{i=1}^{k-1}\sum_{j=1}^{k} a_{ij} \leq \sum_{i=1}^{k-1} y = (k-1)y.$$

It follows that for some $t \leq k-1$ we have $\Sigma_{i=1}^{k-1} a_{it} \leq y$, and for $i = 1, \cdots, k-1$, we clearly have $\Sigma_{j=1;j\neq t}^{k} a_{ij} \leq \Sigma_{j=1}^{k} a_{ij} \leq y$. Thus

$$\left\{\prod_{m=1}^{k-1} r_1(a_{mt})\right\}\prod_{i=1}^{k-1}\left\{\prod_{j=1;j\neq t}^{k} r_1(a_{ij})\right\}$$

is a term of $(R_{k-1}(y))^k$. Hence each term of $(R_k(y))^{k-1}$ occurs as a term of $(R_{k-1}(y))^k$. However, since t could have any of $k-1$ different values, it follows that we may have associated as many as, but no more than, $k-1$ different terms of $(R_k(y))^{k-1}$ with the same term of $(R_{k-1}(y))^k$. Therefore we have

$$(R_k(y))^{k-1} \leq (k-1)(R_{k-1}(y))^k. \tag{8}$$

This completes the proof of Theorem 1.

The constant $k-1$ in (8) is probably not best possible. The correct constant is most likely 1, but we have only been able to prove this in certain special cases. For example, if $r_1(n) = 1$ for all n, then

$$\begin{aligned}(R_k(n))^{k-1} &= ((n+1)(n+2)\cdots(n+k)/k!)^{k-1}\\ &= (1+n/1)^{k-1}(1+n/2)^{k-1}\cdots(1+n/k)^{k-1}\\ &\leq (1+n/1)^k(1+n/2)^k\cdots(1+n/(k-1))^k\\ &= ((n+1)(n+2)\cdots(n+k-1)/(k-1)!)^k\\ &= (R_{k-1}(n))^k.\end{aligned}$$

Taking $\delta = 1-\beta$ in Theorem 1, we obtain the following

Corollary. Let $0 < \beta < 1$. *If* $R_k(n) = G(n) + v(n)$ *with* $v(n) = o(G(n))$, *if* $G(n) \sim cn^\beta$ *with* $c > 0$, *and if* $\Delta^2 G(n) \leq 0$ *for all sufficiently large* n, *then for every* $\epsilon > 0$,

$$v(n) \neq o(n^{\beta(1-\beta)(1-1/k)/(1-\beta+\beta/k)-\epsilon}).$$

In our second theorem, we prove that even if very little is known about the exact order of magnitude of $G(n)$, we can still claim that $v(n) \neq O(1)$.

Theorem 2. *If* $R_k(n) = G(n) + v(n)$ *with* $v(n) = o(G(n))$, *if* $G(n) = o(n)$, *but* $G(n) \to \infty$ *as* $n \to \infty$, *and if* $\Delta^2 G(n) \leq 0$ *for all sufficiently large* n, *then* $v(n) \neq O(1)$.

Proof. Suppose there exists $M > 0$ such that $|v(n)| \leq M$ for all n. Recall that in the proof of Theorem 1, we showed that $\Delta G(n) << G(n)/n$ is a consequence of the hypothesis $\Delta^2 G(n) \leq 0$ for all sufficiently large n. Since we also have $G(n) = o(n)$, we conclude $\Delta G(n) = o(1)$. The hypotheses $R_k(n) \sim G(n)$ and $G(n) \to \infty$ as $n \to \infty$ guarantee that we have $r_1(n) \neq 0$, and hence $r_1(n) \geq 1$, for infinitely many n. Thus there exist integers a and A such that $R_1(A) - R_1(a) \geq 2M+1$. Choose N so that $r_1(N) \neq 0$. Since $R_1(N) \geq 1$, we have

$$2M+1 \leq R_1(A) - R_1(a) = \sum_{a<n\leq A} r_1(n) \leq \sum_{a<n\leq A} r_1(n)(r_1(N))^{k-1}.$$

Using (1) and (2), we see that every term of the last sum is a term of $R_k((k-1)N+A) - R_k((k-1)N+a)$. Since N can be chosen arbitrarily large, we therefore have

$$\begin{aligned} 2M + 1 &\le R_k((k-1)N + A) - R_k((k-1)N + a) \\ &= G((k-1)N + A) - G((k-1)N + a) + v((k-1)N + A) - v((k-1)N + a) \\ &\le (A - a)\Delta G((k-1)N + a) + 2M \\ &= o(1) + 2M \quad \text{as } N \longrightarrow \infty. \end{aligned}$$

The assumption $v(n) = O(1)$ has led us to a contradiction. Therefore we conclude $v(n) \neq O(1)$.

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DEPARTMENT OF MATHEMATICS, WAKE FOREST UNIVERSITY, WINSTON-SALEM, NORTH CAROLINA 27109