AN ELEMENTARY METHOD FOR ESTIMATING ERROR TERMS IN ADDITIVE NUMBER THEORY 1

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ABSTRACT. Let $R_k(n)$ denote the number of ways of representing the integers not exceeding $n$ as the sum of $k$ members of a given sequence of nonnegative integers. Using only elementary methods, we prove a general theorem from which we deduce that, for every $\varepsilon > 0$,

$$R_k(n) = cn^\beta \neq o(n^{\beta(1-\beta)(1-1/k)/(1-\beta+\beta/k)-\varepsilon})$$

where $c$ is a positive constant and $0 < \beta < 1$.

Let $R_k(n)$ denote the number of ways of representing the integers not exceeding $n$ as the sum of $k$ members of a given sequence of nonnegative integers. Jurkat [4] has shown that $R_k(n) = G(n) = o(n^{\beta/4})$ whenever $k$ is an even integer, $0 < \beta < 2$, and $G(n)$ is a logarithmico-exponential function with $G(n) \sim cn^\beta$, $c > 0$. Randol [5] has shown that $R_k(n) = cn^\beta = o(n^{\beta(1-1/k)(1-\beta/k)})$ when $m = k/\beta$ is an even integer, the given sequence of nonnegative integers is the sequence $\{n^m\}_{n=1}^\infty$, and $c$ is the volume of the $k$-dimensional solid defined by $y_1^m + y_2^m + \cdots + y_k^m \leq 1$. The corollary to our first theorem improves Jurkat's result in case $\beta < (3k-4)/(3k-3)$ and comes surprisingly close to Randol's result even though Randol's theorem deals only with a very special case of ours. In contrast to the methods employed by others on this type of problem (see [1]—[6]), the techniques we use here are all elementary.

We begin by defining our notation. Let $r_1(n)_{n=0}^\infty$ be a sequence of nonnegative real numbers such that if $r_1(n) \neq 0$, then $r_1(n) \geq 1$. (The lower bound 1 is chosen for convenience; any positive lower bound would suffice.) If $k$ is an integer, $k \geq 2$, define $r_k(n)$ by

$$r_k(n) = \sum_{m_1 + \cdots + m_k = n} r_1(m_1) \cdots r_1(m_k) = \sum_{m=0}^{n} r_1(m)r_{k-1}(n-m).$$

$R_k(n)$ will denote the summatory function of $r_k(n)$. Thus

$$R_k(n) = \sum_{m=0}^{n} r_k(m).$$

If $r_1(n)$ is a nonnegative integer for all $n$, we can interpret $r_1(n)$ as the number of occurrences of $n$ in a given sequence $\{a_m\}$ of nonnegative integers.
In this case, \( r_k(n) \) denotes the number of ways \( n \) can be represented as the sum of \( k \) elements of the sequence \( \{a_n\} \). It is interesting to note, however, that the proofs of our theorems do not require \( r_k(n) \) to be integer-valued.

Finally, define \( \Delta G(n) = G(n) - G(n - 1) \) and \( \Delta^2 G(n) = \Delta(\Delta G(n)) \). We write \( f(n) \ll g(n) \) when \( f(n) \) is less than a positive multiple of \( g(n) \) for all sufficiently large \( n \). Our main result is

**Theorem 1.** Let \( 0 < \beta < 1 - \delta < 1 \). If \( R_k(n) = G(n) + v(n) \), with \( v(n) = o(G(n)) \), if \( n^\beta \ll G(n) \ll n^{1-\delta} \), and if \( \Delta^2 G(n) \leq 0 \) for all sufficiently large \( n \), then for every \( \epsilon > 0 \) we have

\[
v(n) \ll o(n^{\beta(1-1/\epsilon)/(1-\beta+\beta/\epsilon)-\epsilon}).
\]

**Proof.** By hypothesis, there exists \( n_0 > 0 \) such that \( \Delta^2 G(n) \leq 0 \) for \( n \geq n_0 \), i.e., \( \Delta G(n) \) is nonincreasing for \( n \geq n_0 \). Thus for \( n \geq n_0 \) we have

\[
(n - n_0)\Delta G(n) \leq \sum_{m=n_0+1}^{n} \Delta G(m) = G(n) - G(n_0),
\]

and hence

\[
\Delta G(n) \ll G(n)/n.
\]

Choose \( x \in \mathbb{Z}^+ \) such that \( r_1(x+1) \neq 0 \) and assume \( 2 < y < x \), where \( y \) is an integer to be specified later. Since \( R_k(n) \sim G(n) \) and \( G(n) \to \infty \) as \( n \to \infty \), there exist arbitrarily large \( n \) for which \( r_1(n) \neq 0 \). Hence \( x \) can be taken arbitrarily large. Using (2) and (1), we see that

\[
R_k(x+y) - R_k(x) = \sum_{m=x+1}^{x+y} r_k(m) = \sum_{m=x+1}^{x+y} \sum_{j=0}^{m} r_1(j) r_{k-1}(m-j) \geq r_1(x+1) \sum_{m=0}^{y-1} r_{k-1}(m) = r_1(x+1) R_{k-1}(y-1).
\]

At first glance, the above estimate might seem to be rather crude. However, since by hypothesis \( G(n) = o(n) \), and hence \( R_k(n) = o(n) \), it follows that it is very unlikely that \( r_1(n) \neq 0 \) for very many \( n \) between \( x \) and \( 2x \). Thus the above estimate is good when \( R_k(n) \) is significantly smaller than \( n \) in magnitude.

For the moment, let us assume that we know

\[
R_{k-1}(y) \gg (R_k(y))^{1-1/k}.
\]

Now using \( G(n) \gg n^\beta \) and recalling \( R_k(n) \sim G(n) \), we obtain, for sufficiently large \( y \),

\[
R_k(x+y) - R_k(x) \gg (R_k(y-1))^{1-1/k} \gg (G(y-1))^{1-1/k} \gg y^{\beta(1-1/k)}.
\]

On the other hand, if \( v(n) = o(n^\alpha) \), then using the fact that \( \Delta G(n) \) is nonin-
creasing and applying (4), we get
\[ R_k(x + y) - R_k(x) = G(x + y) - G(x) + v(x + y) - v(x) \]
\[ \leq y \Delta G(x) + o(x^\alpha) \ll yG(x)/x + o(x^\alpha). \]
Thus we have, for sufficiently large \( y \),
\[ y^\beta(1-1/k) \ll yG(x)/x + o(x^\alpha). \]
We now choose \( y \) so that
\[ yG(x)/x = o(y^\beta(1-1/k)), \]
say
\[ y = [(x/G(x))^{1/(1-\beta+\beta/k)}-\epsilon], \quad \epsilon > 0. \]
It is easily verified that \( y < x \), and when \( \epsilon \) is small, \( y \) grows large with \( x \).
With our choice of \( y \), we see that a contradiction of (6) will occur if \( x < \delta(1-1/k)/(1-\beta + \beta/k) \), since then for sufficiently small \( \epsilon \) we have
\[ x^\alpha \ll (x/G(x))^{\beta(1-1/k)}(1/(1-\beta+\beta/k)-\epsilon) \]
\[ \ll y^\beta(1-1/k)(1/(1-\beta+\beta/k)-\epsilon) \ll y^\beta(1-1/k). \]
The proof of the theorem will be complete if we can verify (5). In fact we shall show
\[ (R_k(y))^{k-1} \leq (k-1)(R_{k-1}(y))^k. \]
We write
\[ (R_k(y))^{k-1} = \left( \sum_{a_1 + \cdots + a_k = y} r_1(a_1) \cdots r_1(a_k) \right)^{k-1}. \]
If we multiply out the right side of the last equation, we see that a typical term of \((R_k(y))^{k-1}\) is \( \Pi_{i=1}^{k-1} \Pi_{j=1}^{k} r_1(a_{ij}) \), where \( \Sigma_{j=1}^{k} a_{ij} \leq y \) for \( i = 1, 2, \ldots, k-1 \). Now
\[ \sum_{i=1}^{k-1} \sum_{j=1}^{k} a_{ij} \leq \sum_{i=1}^{k-1} y = (k-1)y. \]
It follows that for some \( t \leq k-1 \) we have \( \Sigma_{i=1}^{k-1} a_{it} \leq y \), and for \( i = 1, \ldots, k-1 \), we clearly have \( \Sigma_{j=1; j \neq t}^{k} a_{ij} \leq \Sigma_{j=1}^{k} a_{ij} \leq y \). Thus
\[ \left\{ \prod_{m=1}^{k-1} r_1(a_{mt}) \right\} \prod_{i=1}^{k-1} \left\{ \prod_{j=1; j \neq t}^{k} r_1(a_{ij}) \right\} \]
is a term of \((R_{k-1}(y))^k\). Hence each term of \((R_k(y))^{k-1}\) occurs as a term of \((R_{k-1}(y))^k\). However, since \( t \) could have any of \( k-1 \) different values, it follows that we may have associated as many as, but no more than, \( k-1 \) different terms of \((R_k(y))^{k-1}\) with the same term of \((R_{k-1}(y))^k\). Therefore we have
This completes the proof of Theorem 1.

The constant $k - 1$ in (8) is probably not best possible. The correct constant is most likely 1, but we have only been able to prove this in certain special cases. For example, if $r_1(n) = 1$ for all $n$, then

\[(R_k(n))^{k-1} = ((n + 1)(n + 2) \cdots (n + k)/k)^{k-1}\]
\[= (1 + n/1)^{k-1}(1 + n/2)^{k-1} \cdots (1 + n/k)^{k-1}\]
\[\leq (1 + n/1)^k(1 + n/2)^k \cdots (1 + n/(k - 1))^k\]
\[= ((n + 1)(n + 2) \cdots (n + k - 1)/(k - 1))^k\]
\[= (R_{k-1}(n))^k.\]

Taking $\delta = 1 - \beta$ in Theorem 1, we obtain the following

Corollary. Let $0 < \beta < 1$. If $R_k(n) = G(n) + v(n)$ with $v(n) = o(G(n))$, if $G(n) \sim cn^\beta$ with $c > 0$, and if $\Delta^2 G(n) \leq 0$ for all sufficiently large $n$, then for every $\epsilon > 0$,

\[v(n) \neq o(n^{\beta(1 - \beta)(1 - 1/\epsilon)}(1 - \beta + \beta /k - \epsilon)).\]

In our second theorem, we prove that even if very little is known about the exact order of magnitude of $G(n)$, we can still claim that $v(n) \neq O(1)$.

Theorem 2. If $R_k(n) = G(n) + v(n)$ with $v(n) = o(G(n))$, if $G(n) = o(n)$, but $G(n) \to \infty$ as $n \to \infty$, and if $\Delta^2 G(n) \leq 0$ for all sufficiently large $n$, then $v(n) \neq O(1)$.

Proof. Suppose there exists $M > 0$ such that $|v(n)| \leq M$ for all $n$. Recall that in the proof of Theorem 1, we showed that $\Delta G(n) \ll G(n)/n$ is a consequence of the hypothesis $\Delta^2 G(n) \leq 0$ for all sufficiently large $n$. Since we also have $G(n) = o(n)$, we conclude $\Delta G(n) = o(1)$. The hypotheses $R_k(n) \sim G(n)$ and $G(n) \to \infty$ as $n \to \infty$ guarantee that we have $r_1(n) \neq 0$, and hence $r_1(n) \geq 1$, for infinitely many $n$. Thus there exist integers $a$ and $A$ such that $R_1(A) - R_1(a) \geq 2M + 1$. Choose $N$ so that $r_1(n) \neq 0$. Since $R_1(N) \geq 1$, we have

\[2M + 1 \leq R_1(A) - R_1(a) = \sum_{a < n \leq A} r_1(n) \leq \sum_{a < n \leq A} r_1(n)(r_1(n))^{k-1}.\]

Using (1) and (2), we see that every term of the last sum is a term of

$R_k((k - 1)N + a) - R_k((k - 1)N + a)$. Since $N$ can be chosen arbitrarily large, we therefore have
$2M + 1 \leq R_k((k - 1)N + A) - R_k((k - 1)N + a)$

$= G((k - 1)N + A) - G((k - 1)N + a) + \nu((k - 1)N + A) - \nu((k - 1)N + a)$

$\leq (A - a)\Delta G((k - 1)N + a) + 2M$

$= o(1) + 2M$ as $N \to \infty$

The assumption $\nu(n) = O(1)$ has led us to a contradiction. Therefore we conclude $\nu(n) \neq O(1)$.

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