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CERTAIN OVERRINGS OF RIGHT HEREDITARY, RIGHT NOETHERIAN RINGS ARE V-RINGS

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ABSTRACT. It will be shown that a particular quotient ring of a ring R is a nontrivial V-ring if R is a special kind of a right hereditary, right Noetherian ring. Another result states that all overrings of a right and left Goldie V-ring, which is semiartinian modulo every essential right ideal, are V-rings.

The principal results of this paper describe how one can construct nonsemisimple V-rings from certain right hereditary, right Noetherian rings satisfying the restricted minimum condition. Let R be a ring and $\mathbf{T}$ the torsion theory in Mod-R which is generated by all noninjective simple R-modules; then Theorem 2.1 shows that for a right hereditary, right Noetherian ring R with restricted minimum condition and zero socle, the quotient ring of R with respect to $\mathbf{T}$ is injective modulo every essential right ideal. Theorem 2.2 states that all overrings of a right Goldie V-ring R, which is semiartinian modulo every essential right ideal and has a two-sided classical quotient ring, are rings with the same properties. Cozzens' examples ([3], cf. also [11]) are the only known V-domains which are not semisimple. Being principal right and left ideal domains, they have all the properties mentioned above.

All rings are assumed to have a unit element. "Module" means right module. They are called semisimple if they coincide with their socle. By "overring", only those rings contained in the classical quotient ring are meant.

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1. Basic concepts and their properties.

Definition 1.1. A ring R is a V-ring if every simple R-module is injective. Equivalently, every R-module has radical zero [4, p. 130]. A GV-ring ("generalized V-ring") R is a ring whose simple modules are either projective or injective. Equivalently R has radical zero modulo every essential right ideal and $Z_r(R) \cap J(R) = (0)$ [12, p. 71], $Z_r(R)$ denoting the right singular ideal and $J(R)$ the Jacobson radical of R.

V-rings are weakly regular, i.e. every right ideal is idempotent [10, p. 187]. In [8] the following result was proved:

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Proposition 1.2. *Every weakly regular ring with maximum condition for annihilating left ideals is a direct sum of finitely many simple rings. This holds in particular for weakly regular right or left Goldie rings.*

The following proposition is a generalization of "A semiprime right Noetherian GV-ring is a V-ring", found in [12, p. 75].

Proposition 1.3. *Every semiprime GV-ring R with maximum condition for annihilating left ideals is a V-ring (therefore a direct sum of finitely many simple V-rings).*

Proof. By [9, p. 70], R is an irredundant subdirect sum of finitely many prime rings R_i, $i = 1, \ldots, n$, of the form $R_i = R/P_i$, P_i some maximal annihilating ideals of R. An essential right ideal modulo P_i is an essential one in R. As an immediate consequence R_i is a GV-ring, too, which satisfies the maximum condition for annihilating left ideals. According to [12, p. 75] any prime GV-ring is weakly regular, hence R_i, $i = 1, \ldots, n$, is a simple ring because of Proposition 1.2. This implies $R = \bigoplus_{i=1}^{n} R_i$. Any simple GV-ring is a V-ring, for if its socle is zero, it obviously is a V-ring, if the socle is nonzero, the ring is semisimple. Direct sums of finitely many V-rings are also V-rings. This completes the proof.

Corollary 1.4. *Any GV-domain is a simple V-domain.*

Definition 1.5. A ring R satisfies the restricted minimum condition for right ideals, briefly RMR-condition, if R/W is an Artinian R-module for every essential right ideal W of R.

Clearly all modules belonging to the Goldie torsion class of a ring with RMR-condition are semiartinian; a module M is called semiartinian if every nonzero homomorphic image of M contains a simple module.

Each right and left hereditary and Noetherian ring fulfills the RMR-condition [2, p. 84].

In [10, Theorem 4.2] it was shown (slightly altered version) that a ring R is a ring with Krull dimension at most one whose simple modules are injective iff R is a direct sum of finitely many rings S_i, each of which is a right hereditary, right Noetherian simple V-ring with RMR-condition. The proofs of the following lemmata use ideas found in the proof of this theorem.

Lemma 1.6. *A right Noetherian ring R is a GV-ring with RMR-condition iff every R-module belonging to the Goldie torsion class is semisimple and injective. In this case R is right hereditary.*

Proof. Let R be a right Noetherian GV-ring with RMR-condition. Each cyclic module R/W, W an essential right ideal of R, is Artinian and has radical zero (see remark in Definition 1.1), thus is semisimple. It is injective since no simple submodule of R/W can be projective (otherwise there would be a simple projective module of the form $E = V/W$ implying that W is a direct summand of V; this is impossible). The Goldie torsion theory is generated by modules of the form R/W [14, p. 9], and one part of the assertion follows. The other direction is trivial.

To show that R is hereditary, let M be any R-module and E the injective hull of M. Since E/M is injective by the above, this implies that inj dim $M \leq 1$ and, hence, the right injective global dimension of R is ≤ 1, which is equivalent to the assertion.

Lemma 1.7. *Each right ideal $\neq (0)$ of a right Noetherian V-ring with RMR-condition is generated by two elements.*

Proof. It obviously is enough to prove the assertion for essential right ideals W. Now W contains a right ideal cR, c regular. Moreover W/cR is cyclic because it is a direct summand of the (semisimple) module R/cR. Therefore W is generated by two elements.

2. **Simple injective modules, quotient rings and overrings.** First let us summarize some needed properties of a right quotient ring with respect to an hereditary torsion theory $\mathbf{T}$ of a ring R. Throughout, R is assumed to be $\mathbf{T}$-torsion-free. For the proofs and further properties we mainly refer to [7], additionally to [13].

1. The quotient ring of R with respect to $\mathbf{T}$ is denoted by $Q_{\mathbf{T}}(R)$. R can be identified with a subring of $Q_{\mathbf{T}}(R)$, such that $Q_{\mathbf{T}}(R)$ is an essential extension of R_R. $Q_{\mathbf{T}}(R)/R$ is a $\mathbf{T}$-torsion module. Each $\mathbf{T}$-torsion-free, R-injective R-module is an injective $Q_{\mathbf{T}}(R)$-module.

2. The maximal quotient ring $Q_{\mathbf{M}}(R)$ is the quotient ring with respect to the maximal torsion theory $\mathbf{M}$ such that R is torsion-free. $\mathbf{M}$ coincides with the Goldie torsion theory if R is right nonsingular. $Q_{\mathbf{T}}(R) = \{r \in Q_{\mathbf{M}}(R);$ there is a right ideal $L \in F(\mathbf{T})$ with $rL \subseteq R\}$; $F(\mathbf{T})$ denotes the topology corresponding to $\mathbf{T}$.

3. $\mathbf{T}$ has property (T) iff one of the following equivalent conditions holds:

(a) Every $Q_{\mathbf{T}}(R)$-module is $\mathbf{T}$-torsion free and $\mathbf{T}$-injective.

(b) (i) If $L_1 \subseteq L_2 \subseteq L_3 \subseteq \ldots$ is an ascending chain of right ideals, whose union is in $F(\mathbf{T})$, then $L_n \in F(\mathbf{T})$ for some n. (ii) Every right ideal $L \in F(\mathbf{T})$ is $\mathbf{T}$-projective.

In this case every right ideal of $Q_{\mathbf{T}}(R)$ is generated by a right ideal of R.

It is clear that every hereditary torsion theory of a right hereditary, right Noetherian ring has property (T).

4. If $\mathbf{T}$ is such that $Q_{\mathbf{T}}(R)$ is a semisimple (Artinian) ring, then $\mathbf{T} = \mathbf{M}$, i.e. $Q_{\mathbf{T}}(R)$ is the maximal quotient ring of R.

Theorem 2.1. *Let R be a right hereditary, right Noetherian ring with*

RMR-condition and zero socle, $\mathbf{T}$ *the torsion theory generated by all the simple noninjective R-modules. Then* $Q_{\mathbf{T}}(R)$ *is a right hereditary, right Noetherian GV-ring with RMR-condition.* $Q_{\mathbf{T}}(R)$ *is not semisimple iff there is a simple injective R-module. If R additionally is semiprime,* $Q_{\mathbf{T}}(R)$ *is a V-ring.*

Proof. Let S be an essential right ideal of $Q_{\mathbf{T}}(R)$. From the properties of $Q_{\mathbf{T}}(R)$ listed above, it follows that S is an essential R-submodule of $Q_{\mathbf{T}}(R)$, hence $F = Q_{\mathbf{T}}(R)/S$ is a Goldie torsion R-module. As F is $\mathbf{T}$-torsion-free, all its simple R-submodules are injective. Thus the socle of F is also injective; it must coincide with F, since F is semiartinian. Being R-injective and $\mathbf{T}$-torsion free, F is injective over $Q_{\mathbf{T}}(R)$. Thus $Q_{\mathbf{T}}(R)$ is injective modulo every essential right ideal and, in particular, is a GV-ring. $Q_{\mathbf{T}}(R)$ is right Noetherian because each of its right ideals is generated by a right ideal of R. Since an essential right ideal W of $Q_{\mathbf{T}}(R)$ is generated by the essential right ideal $W \cap R$ of R, $Q_{\mathbf{T}}(R)$ satisfies the RMR-condition. Moreover it is right hereditary by Lemma 1.6. Finally $Z_r(R) = (0)$ so that the following argument holds: There is a simple injective R-module iff $\mathbf{T}$ is not the Goldie torsion theory iff $Q_{\mathbf{T}}(R)$ is not the maximal quotient ring iff $Q_{\mathbf{T}}(R)$ is not semisimple. If R is semiprime, $Q_{\mathbf{T}}(R)$ is a V-ring by Proposition 1.3.

Clearly the main part of the theorem remains true if $\mathbf{T}$ is a torsion theory generated by any class of simple modules containing all simple noninjective modules.

The next theorem may be regarded as a continuation of the preceding theorem in constructing V-rings.

Theorem 2.2. *Let R be a right Goldie V-ring which is semiartinian modulo every essential right ideal and let the classical right quotient ring of R be two-sided. Then each overring S of R is a right Goldie V-ring which is semiartinian modulo every essential right ideal.*

Proof. Every essential right ideal L of S obviously is also essential as an R-submodule of S, thus S/L is a semiartinian R-module. Each of its factor modules contains a (simple) divisible R-module, whence R/L is a divisible R-module itself. Each divisible R-submodule N of an S-module M is an S-module, for let $n = \sum_{i=1}^{k} n_i s_i \in NS$, $n_i \in N$, $s_i = a_i^{-1} b_i \in S$; there are $f_i \in N$ such that $f_i a_i = n_i$, whence $n = \sum_{i=1}^{k} f_i b_i \in N$. In particular we have shown that S/L also is semiartinian as an S-module and that each simple S-module E either is projective over S or is simple over R. In the latter case, E is injective over S by the following argument: Let F be an essential right ideal of S, and $h\colon F \to E$ an S-homomorphism. There exists an extension $h'\colon S \to E$ which is R-linear. Let s, $s' = a^{-1}b$ be in S. Since S/F is divisible, s has the form $s = ta + f$, $t \in S$, $f \in F$. Now

$$h'(ss') = h'(tb + fa^{-1}b) = h'(t)b + h(f)a^{-1}b = h'(ta)a^{-1}b + h(f)a^{-1}b$$
$$= h'(ta + f)a^{-1}b = h(s)s'.$$

Thus we have shown that h' is S-linear, and hence E is injective over S. Being a right Goldie GV-ring, S is a V-ring by Proposition 1.3.

It easily can be checked that S is a right hereditary, right Noetherian ring with RMR-condition if R is right Noetherian.

Examples. 1. Let K be any field, $f: K \to K$ a monomorphism and D an f-derivation. The skew polynomial ring in X with $kX = Xf(k) + D(k)$, $k \in K$, is denoted by $R = K[X; f, D]$. There is a natural right algorithm in R, and so it is a principal right ideal domain with RMR-condition. Additionally R is a principal left ideal domain iff f is an automorphism.

In case K is algebraically closed and of characteristic $p > 0$, f the automorphism $k \to k^p$ and $D = 0$, then all cyclic R-modules of the form R/gR are injective, the constant term of $g \in R$ being nonzero. The topology generated by all simple noninjective modules consists of the ideals (X), (X^2), (X^3), $\ldots$. That means, the localization of R at the ideal (X) is a V-ring (see [3] and [11]).

2. As shown in [1] any overring of a principal right and left ideal domain D is a quotient ring with respect to some hereditary torsion theory in Mod-D (in fact the same is true for two-sided hereditary Noetherian prime rings). Here every nontrivial hereditary torsion theory is generated by a class of simple modules. Both examples in [3] have only one isomorphism class of simple modules, i.e. they possess nō proper overrings except the quotient field.

Example (a) in [11, p. 606] is a principal right and left ideal domain which is a V-ring and has an infinite number of isomorphism classes of simple modules, therefore it has nontrivial overrings. They are also principal right and left ideal domains and, by Theorem 2.2, V-rings. (Of course this can be shown by more simple arguments depending on the fact that here overrings are quotient rings.)

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