Sometimes Only Square Matrices Can Be Diagonalized

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Abstract. It is proved that every square matrix over a serial ring is equivalent to some diagonal matrix, even though there are rectangular matrices over these rings which cannot be diagonalized.

Two matrices $A$ and $B$ over a ring $R$ are equivalent if $B = PAQ$ for invertible matrices $P$ and $Q$ over $R$.

Following R. B. Warfield, we will call a module serial if its submodules are totally ordered by inclusion; and we will call a ring $R$ serial if $R$ is a direct sum of serial left modules, and also a direct sum of serial right modules. When $R$ is artinian, these become the generalized uniserial rings of Nakayama [E-G].

Note that, in serial rings, finitely generated ideals do not have to be principal; for example, let $R$ be any full ring of lower triangular matrices over a field. Moreover, it is well known (and easily proved) that if $aR + bR$ is a nonprincipal right ideal in any ring, the $1 \times 2$ matrix $[a \ b]$ is not equivalent to a diagonal matrix. Thus serial rings do not have the property that every rectangular matrix can be diagonalized.

1. Background. Serial rings are clearly semiperfect ($R/\text{rad}R$ is semisimple artinian and idempotents can be lifted modulo $\text{rad}R$).

(1.1) Every indecomposable projective module over a semiperfect ring $R$ is $\cong eR$ for some primitive idempotent $e$ of $R$. (See [M].)

A double-headed arrow $\twoheadrightarrow$ will denote an "onto" homomorphism.

By a projective cover of a module $X$ we mean a minimal epimorphism $\phi: P \twoheadrightarrow X$ with $P$ projective. ("Minimality" means that no submodule other than $P$ itself is mapped onto $X$; equivalently, ker$\phi$ is superfluous in $P$.)

(1.2) (Uniqueness of the projective cover). Let $f: P \twoheadrightarrow X$ be a module epimorphism with $P$ projective, and suppose that $X$ has a projective cover $\phi: P_1 \twoheadrightarrow X$. Then $P = P' \oplus P''$ where $f(P'') = 0$ and $f: P' \twoheadrightarrow X$ is a projective cover. Moreover, there is an isomorphism $\theta: P' \cong P_1$ such that $(f|P') = \phi\theta$. (See [B], 2.3.)

(1.3) Over a semiperfect ring, every finitely generated module has a projective cover [B, 2.1].

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(1.4) If \( \phi_i : P_i \rightarrow X_i \) is a projective cover of \( X_i \) over any ring \( i = 1, \ldots, n \), then
\[
\phi = \bigoplus_{i=1}^{n} \phi_i : \bigoplus_{i=1}^{n} P_i \rightarrow \bigoplus_{i=1}^{n} X_i
\]
is a projective cover of \( \bigoplus X_i \).

Proof. Since \( X_i \) is finitely generated, minimality of \( \phi_i \) shows that \( P_i \) is finitely generated, too. Hence we easily establish: \( \ker \phi_i \) is superfluous in \( P_i \)
\[
\ker \phi_i \subseteq \text{rad } P_i = \bigcap \text{(all maximal submodules of } P_i)\).

(This is more or less [BO, p. 64, Corollary 3].) Then use the fact that \( \text{rad}(\bigoplus P_i) = \bigoplus \text{rad } P_i \) [BO, p. 64, Corollary 2].

2. Theorem. Let \( \alpha : P \rightarrow Q \) be a homomorphism of finitely generated, projective right modules \( P \) and \( Q \) over a serial ring \( R \). Then there exist decompositions \( P = P_1 \oplus \cdots \oplus P_s \) and \( Q = Q_1 \oplus \cdots \oplus Q_t \) with each \( P_i \) and \( Q_i \) indecomposable, such that
\[
\alpha(P_i) \subseteq Q_i \quad \text{for } i \leq w \quad \text{and} \quad \alpha(P_i) = 0 \quad \text{for } i > w
\]
where \( w = \min\{s, t\} \).

Proof. This theorem is a slight extension of Warfield's theorem [W, 3.3], which states that if \( X \) is a finitely generated submodule of a finitely generated projective module \( Q \) over a serial ring \( R \), then there is a decomposition
\[
Q = Q_1 \oplus \cdots \oplus Q_t \quad \text{(each } Q_i \text{ indecomposable)}
\]
such that
\[
X = X_1 \oplus \cdots \oplus X_t \quad \text{(each } X_i = X \cap Q_i)\).
\]
Choose the numbering so that \( X_i \) is nonzero for \( i \leq w \) and \( X_i = 0 \) for \( i > w \).

Now let \( P \) be the module in our theorem, and \( X = \alpha(P) \). Each \( X_i \) in (3) is finitely generated, because it is a homomorphic image of the finitely generated module \( X \), and therefore has a projective cover \( \phi_i : P_i \rightarrow X_i \). Moreover, \( X_i \) is serial \( (X_i \subseteq \text{the serial module } Q_i) \) and therefore \( P_i \) is indecomposable. (If \( P_i = A \oplus B \), then \( X_i = \phi_i(A) + \phi_i(B) \). Since \( X_i \) is serial we can suppose \( \phi_i(A) \supseteq \phi_i(B) \). Then \( X_i = \phi_i(A) \) and by "minimality" of the projective cover, \( A = P_i \).)

By (1.4),
\[
\bigoplus_{i=1}^{w} \phi_i : \bigoplus_{i=1}^{w} P_i \rightarrow \bigoplus_{i=1}^{w} X_i
\]
is a projective cover of $X$; and by uniqueness of the projective cover (1.2), we can suppose that $\bigoplus P_i$ is a direct summand of $P$ (each $\phi_i$ then becomes the restriction of $a$ to $P_i$) with the complimentary summand being mapped by $a$ to 0.

Thus the theorem is proved except that the $w$ obtained above might be $< \min\{s, t\}$. But, when $i > w$ we have obtained, $\alpha(P_i) = 0 \subseteq Q_i$ provided, of course, that $i \leq \min\{s, t\}$. Thus we can change to $w = \min\{s, t\}$. Q.E.D.

3. Main Theorem. Every square matrix over a serial ring is equivalent to some diagonal matrix.

Proof. Since $R$ is serial, it has a decomposition

\begin{equation}
R = e_1R \oplus \ldots \oplus e_dR
\end{equation}

with $\{e_i\}$ an orthogonal set of primitive idempotents.

Let $\alpha: R^{(n)} \rightarrow R^{(n)}$ be left multiplication by the given $n \times n$ matrix $A$, elements of $R^{(n)}$ being written as columns. It will suffice to write $R^{(n)} = R'_1 \oplus \ldots \oplus R'_n = R_1 \oplus \ldots \oplus R_n$, with each $R'_i$ and $R_i \cong R_R$, and each $\alpha(R'_i) \subseteq R_i$.

The first step is to write $R^{(n)}$ as a direct sum of indecomposable modules $P_k$ and $Q_k$ in such a way that $\alpha$ takes the form

\begin{equation}
\alpha: R^{(n)} = \bigoplus_{k=1}^{nd} P_k \rightarrow R^{(n)} = \bigoplus_{k=1}^{nd} Q_k \quad \text{with } \alpha(P_k) \subseteq Q_k
\end{equation}

where $d$ is the integer defined in (1). To do this, merely apply Theorem 2, with $P = Q = R^{(n)}$. To see that $k$ runs from 1 through $nd$, as claimed in (2), note that the ring of endomorphisms of $e_iR$ is $e_iRe_i$ local since $R$ is semi-perfect [L, p. 76, Proposition 2], and therefore the Azumaya-Krull-Schmidt theorem applies to decompositions of $R^{(n)}$ (see [L, p. 78, Corollary]).

We will have our submodules $R'_i$ and $R_i$ if we can express the set $\{1, 2, \ldots, nd\}$ as the union of $n$ disjoint subsets $S(1), \ldots, S(n)$ such that, for each $b$,

\begin{equation}
\bigoplus_{k \in S(b)} P_k \cong R \quad \text{and} \quad \bigoplus_{k \in S(b)} Q_k \cong R.
\end{equation}

To do this, recall that

\begin{equation}
\bigoplus_{k=1}^{nd} P_k = R^{(n)} = \left( \bigoplus_{i=1}^d e_iR \right)^{(n)}
\end{equation}

By the Krull-Schmidt theorem, there are at least $n$ indices $k$ such that $P_k \cong e_1R$. (There will be more than $n$ such indices $k$ if, say, $e_1R \cong e_2R$.) Choose $n$ of these indices, and write $P_k \cong e_1R$ to indicate that $P_k$ is one of the "chosen" $P$'s which are $\cong e_1R$. Then choose $n$ different $P_k$'s
"$\cong$" $e_i R$, and so on, until we partition $\{P_k\}$ into $d$ disjoint $n$-element subsets, each associated with a distinct $e_i R$ (and thus we exhaust the $e_i R$, too). Similarly designate $n$ of the $Q_k$'s "$\cong"$ each $e_i R$.

We can keep track of all this by introducing a set of ordered triples $(k, i \rightarrow j)$, one for each $k \in \{1, 2, \ldots, nd\}$, to indicate that $P_k "$\cong"$ $e_i R$ and $Q_k "$\cong"$ $e_i R$. The arrow serves as a reminder that we are thinking of $\alpha: P_k \rightarrow Q_k$ as a map: $e_i R \rightarrow e_j R$.

Next, let $M$ be the $d \times d$ matrix of nonnegative integers whose $(i, j)$ entry is

$$m_{ij} = \text{the number of triples of the form } (k, i \rightarrow j) \quad (k \text{ varying}).$$

This matrix $M$ has two properties which interest us. For each $i$, the sum of the entries $\sum_{j=1}^{d} m_{ij}$ in row $i$ equals the number of $P_k$ which are "$\cong" e_i R$; that is, $n$. Similarly the sum of the entries in each column equals $n$.

Now, a theorem of Birkhoff states that if a square matrix $M$ of nonnegative real numbers has all of its row sums and column sums equal to the same number, then $M$ is a sum $u_1M_1 + u_2M_2 + \cdots$ of nonnegative multiples of permutation matrices $M_b$. (A $d \times d$ permutation matrix is a square matrix which can be obtained by permuting the rows of the $d \times d$ identity matrix.) In fact, one can see by looking through the proof of Birkhoff's theorem given in [H, p. 52], if the entries of $M$ are all nonnegative integers, we can choose the multipliers $u_b$ to be nonnegative integers, too. Writing this multiplication as repeated addition, we see that our matrix $M$ defined in (4) is a sum of permutation matrices. Checking the row sums then shows that $M$ is the sum of exactly $n$ permutation matrices

$$(5) \quad M = M_1 + \cdots + M_n \quad (M_b = \text{permutation matrix}).$$

Now we build the sets $S(b)$. For each ordered pair $(i, j)$ such that $m_{ij} \neq 0$, the number of $k$'s which occur in a triple $(k, i \rightarrow j)$ equals $m_{ij}$, and this also equals the number of matrices $M_b$ in (5) whose $(i, j)$ entry equals 1. Therefore we can choose a 1-1 correspondence between the numbers $k$ in the set $\{1, 2, \ldots, nd\}$ and the nonzero entries of the matrices $\{M_b\}$, such that $k \leftrightarrow (i, j)_b = \text{the } (i, j) \text{ entry of } M_b \text{ implies } P_k \cong e_i R$ and $Q_k \cong e_j R$. Then let $S(b)$ be the set of all $k$'s which correspond to an entry of $M_b$, that is, $k \leftrightarrow (_b)$. Since a permutation matrix contains exactly one nonzero entry in each row and one in each column, (3) holds, and the proof is complete.

4. Remark. We have not dealt with uniqueness of the diagonal matrices obtained here because, as observed in [L-R, 4.3], if $R$ is any semiperfect ring, diag$(c_1, \ldots, c_n)$ is equivalent to diag$(d_1, \ldots, d_n) \cong \bigoplus R/c_i R \cong \bigoplus R/d_i R$ (isomorphism of $R$-modules).
REFERENCES


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