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SOMETIMES ONLY SQUARE MATRICES CAN BE DIAGONALIZED

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ABSTRACT. It is proved that every SQUARE matrix over a serial ring is equivalent to some diagonal matrix, even though there are rectangular matrices over these rings which cannot be diagonalized.

Two matrices A and B over a ring R are *equivalent* if $B = PAQ$ for invertible matrices P and Q over R.

Following R. B. Warfield, we will call a module *serial* if its submodules are totally ordered by inclusion; and we will call a ring R *serial* if R is a direct sum of serial left modules, and also a direct sum of serial right modules. When R is artinian, these become the *generalized uniserial* rings of Nakayama [E-G].

Note that, in serial rings, finitely generated ideals do *not* have to be principal; for example, let R be any full ring of lower triangular matrices over a field. Moreover, it is well known (and easily proved) that if $aR + bR$ is a nonprincipal right ideal in *any* ring, the 1×2 matrix $[a\ b]$ is not equivalent to a diagonal matrix. Thus serial rings do not have the property that every rectangular matrix can be diagonalized.

1. **Background.** Serial rings are clearly *semiperfect* ($R/\mathrm{rad}R$ is semisimple artinian and idempotents can be lifted modulo $\mathrm{rad}R$).

(1.1) *Every indecomposable projective module over a semiperfect ring R is $\cong eR$ for some primitive idempotent e of R.* (*See* [M].)

A double-headed arrow $\twoheadrightarrow$ will denote an "onto" homomorphism.

By a *projective cover* of a module X we mean a *minimal epimorphism* ϕ: $P \twoheadrightarrow X$ with P projective. ("Minimality" means that no submodule other than P itself is mapped *onto* X; equivalently, $\ker f$ is superfluous in P.)

(1.2) (*Uniqueness of the projective cover*). *Let f: $P \twoheadrightarrow X$ be a module epimorphism with P projective, and suppose that X has a projective cover ϕ: $P_1 \rightarrowtail X$. Then $P = P' \oplus P''$ where $f(P'') = 0$ and f: $P' \twoheadrightarrow X$ is a projective cover. Moreover, there is an isomorphism θ: $P' \twoheadrightarrow P_1$ such that $(f|P') = \phi\theta$.* (*See* [B, 2.3].)

(1.3) *Over a semiperfect ring, every finitely generated module has a projective cover* [B, 2.1].

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(1.4) *If* $\phi_i\colon P_i \twoheadrightarrow X_i$ *is a projective cover of* X_i *over any ring* ($i = 1, \ldots, n$), *then*

$$\phi = \bigoplus_{i=1}^{n} \phi_i\colon \bigoplus_{i=1}^{n} P_i \twoheadrightarrow \bigoplus_{i=1}^{n} X_i$$

is a projective cover of $\bigoplus X_i$.

Proof. Since X_i is finitely generated, minimality of ϕ_i shows that P_i is finitely generated, too. Hence we easily establish: $\ker\phi_i$ is superfluous in $P_i \Leftrightarrow$

$$\ker \phi_i \subseteq \operatorname{rad} P_i = \bigcap(\text{all maximal submodules of } P_i).$$

(This is more or less [**BO**, p. 64, Corollary 3].) Then use the fact that $\operatorname{rad}(\bigoplus P_i) = \bigoplus \operatorname{rad} P_i$ [**BO**, p. 64, Corollary 2].

2. **Theorem.** *Let* $\alpha\colon P \to Q$ *be a homomorphism of finitely generated, projective right modules* P *and* Q *over a serial ring* R. *Then there exist decompositions* $P = P_1 \oplus \cdots \oplus P_s$ *and* $Q = Q_1 \oplus \cdots \oplus Q_t$ *with each* P_i *and* Q_i *indecomposable, such that*

$$\alpha(P_i) \subseteq Q_i \quad \text{for } i \leq w \quad \text{and} \quad \alpha(P_i) = 0 \quad \text{for } i > w \tag{1}$$

where $w = \min\{s, t\}$.

Proof. This theorem is a slight extension of Warfield's theorem [**W**, 3.3], which states that if X is a finitely generated submodule of a finitely generated projective module Q over a serial ring R, then there is a decomposition

$$Q = Q_1 \oplus \cdots \oplus Q_t \quad (\text{each } Q_i \text{ indecomposable}) \tag{2}$$

such that

$$X = X_1 \oplus \cdots \oplus X_t \quad (\text{each } X_i = X \cap Q_i). \tag{3}$$

Choose the numbering so that X_i is nonzero for $i \leq w$ and $X_i = 0$ for $i > w$.

Now let P be the module in our theorem, and $X = \alpha(P)$. Each X_i in (3) is finitely generated, because it is a homomorphic image of the finitely generated module X, and therefore has a projective cover $\phi_i\colon P_i \twoheadrightarrow X_i$. Moreover, X_i is serial ($X_i \subseteq$ the serial module Q_i) and therefore P_i is indecomposable. (If $P_i = A \oplus B$, then $X_i = \phi_i(A) + \phi_i(B)$. Since X_i is serial we can suppose $\phi_i(A) \supseteq \phi_i(B)$. Then $X_i = \phi_i(A)$ and by "minimality" of the projective cover, $A = P_i$.)

By (1.4),

$$\bigoplus_{i=1}^{w} \phi_i\colon \bigoplus_{i=1}^{w} P_i \twoheadrightarrow \bigoplus_{i=1}^{w} X_i$$

is a projective cover of X; and by uniqueness of the projective cover (1.2), we can suppose that $\bigoplus_i P_i$ is a direct summand of P (each ϕ_i then becomes the restriction of α to P_i) with the complimentary summand being mapped by α to 0.

Thus the theorem is proved except that the w obtained above might be $< \min\{s, t\}$. But, when $i >$ the w we have obtained, $\alpha(P_i) = \{0\} \subseteq Q_i$ *provided,* of course, that $i \leq \min\{s, t\}$. Thus we can change to $w = \min\{s, t\}$. Q.E.D.

3. **Main Theorem.** *Every square matrix over a serial ring is equivalent to some diagonal matrix.*

Proof. Since R is serial, it has a decomposition

$$R = e_1 R \oplus \cdots \oplus e_d R \tag{1}$$

with $\{e_i\}$ an orthogonal set of primitive idempotents.

Let $\alpha: R^{(n)} \to R^{(n)}$ be left multiplication by the given $n \times n$ matrix A, elements of $R^{(n)}$ being written as columns. It will suffice to write $R^{(n)} = R'_1 \oplus \cdots \oplus R'_n = R_1 \oplus \cdots \oplus R_n$, with each R'_i and $R_i \cong R_R$, and each $\alpha(R'_i) \subseteq R_i$.

The first step is to write $R^{(n)}$ as a direct sum of indecomposable modules P_k and Q_k in such a way that α takes the form

$$\alpha: R^{(n)} = \bigoplus_{k=1}^{nd} P_k \to R^{(n)} = \bigoplus_{k=1}^{nd} Q_k \quad \text{with } \alpha(P_k) \subseteq Q_k \tag{2}$$

where d is the integer defined in (1). To do this, merely apply Theorem 2, with $P = Q = R^{(n)}$. To see that k runs from 1 through nd, as claimed in (2), note that the ring of endomorphisms of $e_i R$ is $e_i R e_i$ = local since R is semi-perfect [L, p. 76, Proposition 2], and therefore the Azumaya-Krull-Schmidt theorem applies to decompositions of $R^{(n)}$ (see [L, p. 78, Corollary]).

We will have our submodules R'_i and R_i if we can express the set $\{1, 2, \ldots, nd\}$ as the union of n disjoint subsets $S(1), \ldots, S(n)$ such that, for each b,

$$\bigoplus_{k \in S(b)} P_k \cong R \quad \text{and} \quad \bigoplus_{k \in S(b)} Q_k \cong R. \tag{3}$$

To do this, recall that

$$\bigoplus_{k=1}^{nd} P_k = R^{(n)} = \left(\bigoplus_{i=1}^{d} e_i R \right)^{(n)}$$

By the Krull-Schmidt theorem, there are at least n indices k such that $P_k \cong e_1 R$. (There will be more than n such indices k if, say, $e_1 R \cong e_2 R$.) Choose n of these indices, and write P_k "$\cong$" $e_1 R$ to indicate that P_k is one of the "chosen" P's which are $\cong e_1 R$. Then choose n *different* P_k's

"$\cong$" e_2R, and so on, until we partition $\{P_k\}$ into d disjoint n-element subsets, each associated with a distinct e_iR (and thus we exhaust the e_iR, too). Similarly designate n of the Q_k's "$\cong$" each e_iR.

We can keep track of all this by introducing a set of ordered triples $(k, i \to j)$, one for each $k \in \{1, 2, \ldots, nd\}$, to indicate that P_k "$\cong$" e_iR and Q_k "$\cong$" e_jR. The arrow serves as a reminder that we are thinking of α: $P_k \to Q_k$ as a map: $e_iR \to e_jR$.

Next, let M be the $d \times d$ matrix of nonnegative integers whose (i, j) entry is

$$(4) \qquad m_{ij} = \text{the number of triples of the form } (k, i \to j) \qquad (k \text{ varying}).$$

This matrix M has two properties which interest us. For each i, the sum of the entries $\sum_{j=1}^{d} m_{ij}$ in row i equals the number of P_k which are "$\cong$" e_iR; that is, n. Similarly the sum of the entries in each column equals n.

Now, a theorem of Birkhoff states that if a square matrix M of nonnegative real numbers has all of its row sums and column sums equal to the same number, then M is a sum $u_1M_1 + u_2M_2 + \cdots$ of nonnegative multiples of permutation matrices M_b. (A $d \times d$ *permutation matrix* is a square matrix which can be obtained by permuting the rows of the $d \times d$ identity matrix.) In fact, one can see by looking through the proof of Birkhoff's theorem given in [**H**, p. 52], if the entries of M are all nonnegative *integers*, we can choose the multipliers u_b to be nonnegative *integers*, too. Writing this multiplication as repeated addition, we see that our matrix M defined in (4) is a sum of permutation matrices. Checking the row sums then shows that M is the sum of exactly n permutation matrices

$$(5) \qquad M = M_1 + \cdots + M_n \ (M_b = \text{permutation matrix}).$$

Now we build the sets $S(b)$. For each ordered pair (i, j) such that $m_{ij} \neq 0$, the number of k's which occur in a triple $(k, i \to j)$ equals m_{ij}, and this also equals the number of matrices M_b in (5) whose (i, j) entry equals 1. Therefore we can choose a 1-1 correspondence between the numbers k in the set $\{1, 2, \ldots, nd\}$ and the nonzero entries of the matrices $\{M_b\}$, such that $k \leftrightarrow (i, j)_b$ = the (i, j) entry of M_b implies $P_k \cong e_iR$ and $Q_k \cong e_jR$. Then let $S(b)$ be the set of all k's which correspond to an entry of M_b, that is, $k \leftrightarrow (\ ,\ )_b$. Since a permutation matrix contains exactly one nonzero entry in each row and one in each column, (3) holds, and the proof is complete.

4. **Remark.** We have not dealt with uniqueness of the diagonal matrices obtained here because, as observed in [**L–R**, 4.3], if R is any semiperfect ring, $\operatorname{diag}(c_1, \ldots, c_n)$ is equivalent to $\operatorname{diag}(d_1, \ldots, d_n) \Leftrightarrow \bigoplus R/c_iR \cong \bigoplus R/d_iR$ (isomorphism of R-modules).

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