A PROOF OF THE HOMEOMORPHISM OF LEBESGUE-STIELTJES MEASURE WITH LEBESGUE MEASURE

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ABSTRACT. An elementary proof is given of the fact that nonatomic measures on $n$ space, for which open sets have positive measure, are "homeomorphic". The proof is based on the fact that for such measures all hyperplanes in most directions have measure zero.

It is a known fact, although not well enough known, that each nonatomic Lebesgue-Stieltjes measure, for which open sets have positive measure, is homeomorphic with Lebesgue measure. A proof was given by Oxtoby and Ulam [1] who credited von Neumann with having given the first proof, unpublished and different from theirs. The purpose of this note is to present an easy proof of the theorem.

Let $Q$ be the closed unit $n$ cube. A measure $\mu$ on $Q$ is called a Lebesgue-Stieltjes measure if the Borel sets in $Q$ are measurable.

Theorem. If $\lambda$ and $\mu$ are nonatomic Lebesgue-Stieltjes measures on $Q$ such that $\lambda(Q) = \mu(Q) = 1$, $\lambda(\partial Q) = \mu(\partial Q) = 0$, and for each open $G \subset Q$, $\lambda(G) > 0$ and $\mu(G) > 0$, there is a homeomorphism $f$ of $Q$ onto itself such that for each $E \subset Q$, $\mu(E) = \lambda[f(E)]$.

We give two lemmas the first of which is our main simplifying idea in the proof of the Theorem; this lemma is known but has not appeared in print.

Lemma 1. If $\mu$ is a totally finite nonatomic Lebesgue-Stieltjes measure, there is a direction $\theta$ such that all hyperplanes in direction $\theta$ have $\mu$ measure zero.

Proof. The set $\Theta$ of all directions is the set of points on the unit $(n-1)$-dimensional hemisphere. We consider $k$ flats in $n$ space. A straight line is a 1 flat, a plane is a 2 flat, and a hyperplane is an $(n-1)$-flat. The set of 1 flats which have positive measure is countable. For each $k = 2, \ldots, n-1$, the set of $k$ flats of positive measure which contain no lower dimensional $j$ flats of positive measure, is countable. For each $k = 1, \ldots, n-1$, let $H_{kr}$, $r = 1, 2, \ldots$, be the countable set of $k$ flats of positive measure which contain no lower dimensional $j$ flats of positive measure. Let $\Theta_{kr}$ be the set of directions of all hyperplanes passing through $H_{kr}$. Each $\Theta_{kr}$ is a set

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of measure 0 on \( \Theta \). So, \( \text{meas}(\bigcup_{k=1}^{n-1} \bigcup_{r=1}^{\infty} \Theta_{kr}) = 0 \), and the lemma is proved.

**Remark.** Since each \( \Theta_{kr} \) is of measure 0 in \( \Theta \), we have actually shown that the set of directions with the property of the lemma is a dense set.

**Lemma 2.** If \( \sigma \) and \( \tau \) are closed \( n \)-cells, \( \lambda \) and \( \mu \) are nonatomic Lebesgue-Stieltjes measures on \( \sigma \) and \( \tau \), respectively, which are positive for open sets, such that \( \lambda(\sigma) = \mu(\tau) < \infty \), \( \lambda(\partial \sigma) = \mu(\partial \tau) = 0 \), and if \( f \) is a homeomorphism of \( \partial \sigma \) on \( \partial \tau \) then, for each \( \epsilon > 0 \), there is a partitioning of \( \sigma \) into non-overlapping \( n \)-cells \( \sigma_1, \ldots, \sigma_r \), each of diameter less than \( \epsilon \), and an extension \( F \) of \( f \) to a homeomorphism of \( \sigma \) on \( \tau \) such that

\[
\lambda(\partial \sigma_i) = \mu[\partial F(\sigma_i)] = 0 \quad \text{and} \quad \lambda(\sigma_i) = \mu[F(\sigma_i)], \quad i = 1, \ldots, r.
\]

**Proof.** Let \( G \) be any extension of \( f \) to a homeomorphism of \( \sigma \) on \( \tau \) and let \( \phi \) be any homeomorphism of \( Q \) on \( \sigma \). The measures \( \lambda \) and \( \mu \) are transferred to \( Q \) by \( \phi^{-1} \) and \( (G \circ \phi)^{-1} \), respectively. We may thus suppose that \( \sigma \) and \( \tau \) are both \( Q \) and \( f: \partial Q \to \partial Q \) is the identity. For then if \( h: Q \to Q \) is the required extension, the composition \( F = G \circ \phi \circ h \circ \phi^{-1} : \sigma \to \tau \) satisfies the conclusion of the lemma for a partitioning \( \sigma_1, \ldots, \sigma_r \) which is the image under \( \phi \) of the partitioning of \( Q \).

We proceed with the argument in two dimensions. It will be seen to extend readily to higher dimensions.

At most countably many sections of \( \sigma = Q \) in each coordinate direction have positive \( \lambda \) measure. Partition \( \sigma \) into cells of diameter less than \( \epsilon \) by means of sections in the coordinate directions with \( \lambda \) measure zero. Denote the cells by \( \sigma_1, \sigma_2, \ldots, \sigma_m \) across the top row, \( \sigma_{m+1}, \ldots, \sigma_{2m} \) across the second row, etc.

The homeomorphism \( h \) will be defined so that the image in \( \tau = Q \) of the top row of cells will be a polygon. The portion of \( \partial \sigma \) the top row of cells includes is, of course, mapped on itself. The lower boundary of the image is a polygonal arc across \( \tau \) which we now describe.

Choose a closed parallelogram \( H \) interior to \( \tau \) with sides in direction \( \theta_1 \) (almost horizontal) and \( \theta_2 \) (almost vertical) given by Lemma 1 and such that the border, \( \tau \setminus H \), satisfies \( \mu(\tau \setminus H) < \eta = \min_i \lambda(\sigma_i) \). This is possible because the directions in Lemma 1 form a dense set.

Let \( l \) denote a segment between the top boundaries of \( H \) and \( \tau \) parallel to the top of \( H \) and extending beyond the (almost) vertical sides of \( H \). Connect its left end by means of two segments in the directions \( \theta_1 \) and \( \theta_2 \), remaining inside \( \tau \setminus H \), with the lower left corner of \( \sigma_1 \). Connect its right end similarly to the lower right corner of \( \sigma_m \). There results an inverted \( U \)-shaped strip inside \( \tau \setminus H \) whose \( \mu \) measure is therefore less than \( \eta \). Now move \( l \) parallel to itself down into \( \tau \), i.e., in direction \( \theta_2 \), forming a family of polygons. As \( l \) moves, the \( \mu \) measure changes continuously because of
the direction in which it moves. The measure of the polygon obtained when \( l \) reaches the bottom of \( H \) is greater than \( 1 - \eta \).

The position of \( l \) at which the measure equals \( \sum_{i=1}^{m} \lambda(\sigma_i) \) forms the image, \( l_1 \), of the top row of cells.

By a completely similar procedure, \( l_1 \) is divided by a polygonal arc from the upper right corner of \( \sigma_1 \) to a point on the lower boundary of \( l_1 \) that cuts off \( \mu \) measure equal to \( \lambda(\sigma_1) \), producing the image of \( \sigma_1 \). Repeating the procedure, the images of \( \sigma_2, \ldots, \sigma_m \) are determined.

The remaining rows of cells are then treated in turn until all the images of the \( \sigma_i \) are specified. Their boundaries have \( \mu \) measure zero since they consist of segments in the directions \( \theta_1, \theta_2 \) and segments on \( \partial \).

The homeomorphism \( h \) may be taken to be any one which carries all cells \( \sigma_i \) onto the image polygons so specified.

Proof of Theorem. For each \( m = 0, 1, 2, \ldots \), let \( f_m \) be a homeomorphism of \( Q \) onto itself with partitionings \( \{\sigma^m_i\} \) and \( \{\tau^m_i\} \) of \( Q \) such that for each \( \sigma^m_i \in \{\sigma^m\} \) there is a \( \tau^m_i \in \{\tau^m\} \) with \( f_m(\sigma^m_i) = \tau^m_i \). Let \( \{\sigma^0\} \) and \( \{\tau^0\} \) consist of \( Q \) itself and let \( f_0 \) be fixed on \( \partial Q \). For each \( m = 1, 2, \ldots \), suppose \( \{\sigma^{m+1}\} \) is a refinement of \( \{\sigma^m\} \) and \( \{\tau^{m+1}\} \) is a refinement of \( \{\tau^m\} \). Moreover, for each \( \sigma^m_i \in \{\sigma^m\} \), all \( f_k, k > m, \) agree with \( f_m \) on \( \partial \sigma^m_i \), and \( \lambda(\partial \sigma^m_i) = 0 \), \( \mu(\partial \sigma^m_i) = 0 \), and \( \lambda(\sigma^m_i) = \mu(\sigma^m_i) \).

For even \( m \), diam \( \sigma^m_i < 1/m \) for each \( \sigma^m_i \in \{\sigma^m\} \) and, for odd \( m \), diam \( \tau^m_i < 1/m \) for each \( \tau^m_i \in \{\tau^m\} \). This is possible with the use of Lemma 2.

We now define the homeomorphism \( F \). For each \( \sigma^m_i \in \{\sigma^m\} \) let \( F = f_m \) on \( \partial \sigma^m_i \). Suppose \( x \) is on no \( \sigma^m_i \) for any \( m \). For each \( m \) there is a \( \sigma^m_i \) with \( x \in \sigma^m_i \). Now \( \lim_{m \to \infty} \text{diam} \sigma^m_i = 0 \), \( \lim_{m \to \infty} \text{diam} f_m(\sigma^m_i) = 0 \), and \( \sigma^m_i \), \( f_m(\sigma^m_i) \) are, respectively, decreasing sequences of compact sets. Let \( F(x) = \bigcap_{m=1}^{\infty} f_m(\sigma^m_i) \). The mapping \( F \) is defined on \( Q \) and is a homeomorphism.

Each open interval \( I \subset Q \) has a unique representation as a countable union of pairwise nonoverlapping cells belonging to the \( \{\sigma^m\}, m = 1, 2, \ldots \). There are finitely many of these cells in each \( \{\sigma^m\} \). Order the cells as \( s_1, s_2, \ldots \). Now, \( \lambda(s_k) = \mu(F(s_k)) \), \( k = 1, 2, \ldots \). It follows that \( \lambda(I) = \mu(F(I)) \) and the theorem is proved.

REFERENCES


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