PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 52, October 1975

A PROOF OF THE HOMEOMORPHISM OF LEBESGUE-STIELTJES MEASURE WITH LEBESGUE MEASURE

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ABSTRACT. An elementary proof is given of the fact that nonatomic measures on n space, for which open sets have positive measure, are "homeomorphic". The proof is based on the fact that for such measures all hyperplanes in most directions have measure zero.

It is a known fact, although not well enough known, that each nonatomic Lebesgue-Stieltjes measure, for which open sets have positive measure, is homeomorphic with Lebesgue measure. A proof was given by Oxtoby and Ulam [1] who credited von Neumann with having given the first proof, unpublished and different from theirs. The purpose of this note is to present an easy proof of the theorem.

Let Q be the closed unit n cube. A measure μ on Q is called a Lebesgue-Stieltjes measure if the Borel sets in Q are measurable.

Theorem. *If λ and μ are nonatomic Lebesgue-Stieltjes measures on Q such that $\lambda(Q) = \mu(Q) = 1$, $\lambda(\partial Q) = \mu(\partial Q) = 0$, and for each open $G \subset Q$, $\lambda(G) > 0$ and $\mu(G) > 0$, there is a homeomorphism f of Q onto itself such that for each $E \subset Q$, $\mu(E) = \lambda[f(E)]$.*

We give two lemmas the first of which is our main simplifying idea in the proof of the Theorem; this lemma is known but has not appeared in print.

Lemma 1. *If μ is a totally finite nonatomic Lebesgue-Stieltjes measure, there is a direction θ such that all hyperplanes in direction θ have μ measure zero.*

Proof. The set Θ of all directions is the set of points on the unit $(n-1)$-dimensional hemisphere. We consider k flats in n space. A straight line is a 1 flat, a plane is a 2 flat, and a hyperplane is an $(n-1)$-flat. The set of 1 flats which have positive measure is countable. For each $k = 2, \ldots, n-1$, the set of k flats of positive measure which contain no lower dimensional j flats of positive measure, is countable. For each $k = 1, \ldots, n-1$, let H_{kr}, $r = 1, 2, \ldots$, be the countable set of k flats of positive measure which contain no lower dimensional j flats of positive measure. Let Θ_{kr} be the set of directions of all hyperplanes passing through H_{kr}. Each Θ_{kr} is a set

Received by the editors June 17, 1974.
AMS (MOS) subject classifications (1970). Primary 28A05.

[1] Supported by a grant from the NSF.

of measure 0 on Θ. So, $\text{meas}(\bigcup_{k=1}^{n-1}\bigcup_{r=1}^{\infty}\Theta_{kr}) = 0$, and the lemma is proved.

Remark. Since each Θ_{kr} is of measure 0 in Θ, we have actually shown that the set of directions with the property of the lemma is a dense set.

Lemma 2. *If σ and τ are closed n-cells, λ and μ are nonatomic Lebesgue-Stieltjes measures on σ and τ, respectively, which are positive for open sets, such that $\lambda(\sigma) = \mu(\tau) < \infty$, $\lambda(\partial\sigma) = \mu(\partial\tau) = 0$, and if f is a homeomorphism of $\partial\sigma$ on $\partial\tau$ then, for each $\epsilon > 0$, there is a partitioning of σ into nonoverlapping n-cells $\sigma_1, \ldots, \sigma_r$, each of diameter less than ϵ, and an extension F of f to a homeomorphism of σ on τ such that $\lambda(\partial\sigma_i) = \mu[\partial F(\sigma_i)] = 0$ and $\lambda(\sigma_i) = \mu[F(\sigma_i)]$, $i = 1, \ldots, r$.*

Proof. Let G be any extension of f to a homeomorphism of σ on τ and let ϕ be any homeomorphism of Q on σ. The measures λ and μ are transferred to Q by ϕ^{-1} and $(G \circ \phi)^{-1}$, respectively. We may thus suppose that σ and τ are both Q and $f\colon \partial Q \to \partial Q$ is the identity. For then if h: $Q \to Q$ is the required extension, the composition $F = G \circ \phi \circ h \circ \phi^{-1}$: $\sigma \to \tau$ satisfies the conclusion of the lemma for a partitioning $\sigma_1, \ldots, \sigma_r$ which is the image under ϕ of the partitioning of Q.

We proceed with the argument in two dimensions. It will be seen to extend readily to higher dimensions.

At most countably many sections of $\sigma = Q$ in each coordinate direction have positive λ measure. Partition σ into cells of diameter less than ϵ by means of sections in the coordinate directions with λ measure zero. Denote the cells by $\sigma_1, \sigma_2, \ldots, \sigma_m$ across the top row, $\sigma_{m+1}, \ldots, \sigma_{2m}$ across the second row, etc.

The homeomorphism h will be defined so that the image in $\tau = Q$ of the top row of cells will be a polygon. The portion of $\partial\sigma$ the top row of cells includes is, of course, mapped on itself. The lower boundary of the image is a polygonal arc across τ which we now describe.

Choose a closed parallelogram H interior to τ with sides in direction θ_1 (almost horizontal) and θ_2 (almost vertical) given by Lemma 1 and such that the border, $\tau\backslash H$, satisfies $\mu(\tau\backslash H) < \eta = \min_i \lambda(\sigma_i)$. This is possible because the directions in Lemma 1 form a dense set.

Let l denote a segment between the top boundaries of H and τ parallel to the top of H and extending beyond the (almost) vertical sides of H. Connect its left end by means of two segments in the directions θ_1 and θ_2, remaining inside $\tau\backslash H$, with the lower left corner of σ_1. Connect its right end similarly to the lower right corner of σ_m. There results an inverted U-shaped strip inside $\tau\backslash H$ whose μ measure is therefore less than η. Now move l parallel to itself down into τ, i.e. in direction θ_2, forming a family of polygons. As l moves, the μ measure changes continuously because of the direction in which it moves. The measure of the polygon obtained when l reaches the bottom of H is greater than $1 - \eta$.

The position of l at which the measure equals $\Sigma_{i=1}^{m} \lambda(\sigma_i)$ forms the image, I_1, of the top row of cells.

By a completely similar procedure, I_1 is divided by a polygonal arc from the upper right corner of σ_1 to a point on the lower boundary of I_1 that cuts off μ measure equal to $\lambda(\sigma_1)$, producing the image of σ_1. Repeating the procedure, the images of $\sigma_2, \ldots, \sigma_m$ are determined.

The remaining rows of cells are then treated in turn until all the images of the σ_i are specified. Their boundaries have μ measure zero since they consist of segments in the directions θ_1, θ_2 and segments on ∂r.

The homeomorphism h may be taken to be any one which carries all cells σ_i onto the image polygons so specified.

Proof of Theorem. For each $m = 0, 1, 2, \ldots$, let f_m be a homeomorphism of Q onto itself with partitionings $\{\sigma^m\}$ and $\{\tau^m\}$ of Q such that for each $\sigma_i^m \in \{\sigma^m\}$ there is a $\tau_i^m \in \{\tau^m\}$ with $f_m(\sigma_i^m) = \tau_i^m$. Let $\{\sigma^0\}$ and $\{\tau^0\}$ consist of Q itself and let f_0 be fixed on ∂Q. For each $m = 1, 2, \ldots$, suppose $\{\sigma^{m+1}\}$ is a refinement of $\{\sigma^m\}$ and $\{\tau^{m+1}\}$ is a refinement of $\{\tau^m\}$. Moreover, for each $\sigma_i^m \in \{\sigma^m\}$, all f_k, $k > m$, agree with f_m on $\partial\sigma_i^m$, and $\lambda(\partial\sigma_i^m) = 0$, $\mu[f_m(\partial\sigma_i^m)] = 0$, and $\lambda(\sigma_i^m) = \mu[f_m(\sigma_i^m)]$. Also for even m, diam $\sigma_i^m < 1/m$ for each $\sigma_i^m \in \{\sigma^m\}$ and, for odd m, diam $\tau_i^m < 1/m$ for each $\tau_i^m \in \{\tau^m\}$. This is possible with the use of Lemma 2.

We now define the homeomorphism F. For each $\sigma_i^m \in \{\sigma^m\}$ let $F = f_m$ on $\partial\sigma_i^m$. Suppose x is on no σ_i^m for any m. For each m there is a $\sigma_{i_m}^m$ with $x \in \sigma_{i_m}^m$. Now $\lim_{m\to\infty}$diam $\sigma_{i_m}^m = 0$, $\lim_{m\to\infty}$diam $f_m(\sigma_{i_m}^m) = 0$, and $\sigma_{i_m}^m$, $f_m(\sigma_{i_m}^m)$ are, respectively, decreasing sequences of compact sets. Let $F(x) = \bigcap_{m=1}^{\infty} f_m(\sigma_{i_m}^m)$. The mapping F is defined on Q and is a homeomorphism.

Each open interval $I \subset Q$ has a unique representation as a countable union of pairwise nonoverlapping cells belonging to the $\{\sigma^m\}$, $m = 1, 2, \ldots$. There are finitely many of these cells in each $\{\sigma^m\}$. Order the cells as s_1, $s_2, \ldots$. Now, $\lambda(s_k) = \mu[F(s_k)]$, $k = 1, 2, \ldots$. It follows that $\lambda(I) = \mu[F(I)]$ and the theorem is proved.

REFERENCES

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