REMARKS ON A COMPARISON THEOREM FOR SCALAR RICCATI EQUATIONS

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ABSTRACT. A comparison theorem for scalar Riccati equations will be stated which contains a recent theorem of Stafford and Heidel.

In a recent paper [1], Stafford and Heidel obtain a comparison theorem for the scalar Riccati equations

\begin{align}
(1) & \quad r'(t) + r^2(t) + q_1(t) = 0, \\
(2) & \quad s'(t) + s^2(t) + q_2(t) = 0.
\end{align}

Theorem (Stafford and Heidel). Suppose that \( q_1 \) and \( q_2 \) are nonnegative on \([a, b)\) where \( 0 < a < b < \infty \) and that

\[ \int_a^t r^2 q_2(r) \, dr \leq \int_a^t r^2 q_1(r) \, dr, \quad t \geq a. \]

If (1) has a solution \( r(t) \) on \([a, b)\) such that \( ar(a) < 1 \) then (2) has a solution on \([a, b)\).

The following theorem is an improvement over the result of Stafford and Heidel.

**Theorem 1.** Suppose that (1) has a solution \( r(t) \) on \([a, b)\) where \( 0 < a < b \leq \infty \), and that there exists a constant \( s_0 \) such that

\[ \int_a^t r^2 q_2(r) \, dr \leq a(1 - ar(a)) + \int_a^t r^2 q_1(r) \, dr. \]

Then (2) has a solution \( s(t) \) on \([a, b)\) such that \( s(a) = s_0 \) and \( |1 - ts(t)| \leq 1 - tr(t) \).

**Proof.** The outline of the proof that we present here follows Hille [2, p. 462]. Consider the integral equations

\begin{align}
(4) & \quad tz(t) = az(a) + \int_a^t z^2(r) \, dr + \int_a^t r^2 q_2(r) \, dr, \\
(5) & \quad ts(t) = a(1 - as_0) + \int_a^t s^2(r) \, dr + \int_a^t r^2 q_1(r) \, dr.
\end{align}
It is easily verified that (4) has the solution \( z(t) = 1 - t r(t) \) on \([a, b)\) which is the limit of a strictly increasing sequence \( \{z_n(t)\} \) of successive approximations satisfying

\[
iz_n(t) = a z(n) + \int_a^t z_{n-1}^2(\tau) d\tau + \int_a^t r^2 q_1(\tau) d\tau,
\]

\[
z_0(t) = a z(a) + \int_a^t r^2 q_1(\tau) d\tau.
\]

If we establish that (5) has a solution \( v(t) \) existing on \([a, b)\), then \( s(t) = (1 - v(t))/t \) is a solution of (2) existing on \([a, b)\) satisfying \( s(a) = s_0 \). A solution of (5) can be constructed as the limit of the sequence \( \{v_n(t)\} \) of solutions to

\[
t v_n(t) = a(1 - s_n) + \int_a^t v_{n-1}^2(\tau) d\tau + \int_a^t r^2 q_2(\tau) d\tau,
\]

\[
v_0(t) = a(1 - s_0) + \int_a^t r^2 q_2(\tau) d\tau.
\]

The convergence of the sequence \( \{v_n(t)\} \) follows from the inequalities

\[
|v_n(t)| \leq z_n(t), \quad |v_{n+1}(t) - v_n(t)| \leq z_{n+1}(t) - z_n(t).
\]

**Corollary 1.** If (1) has a solution \( r(t) \) on \([a, b)\) such that \( ar(a) \leq 1 \) and

\[
\left| \int_a^t r^2 q_2(\tau) d\tau \right| \leq \int_a^t r^2 q_1(\tau) d\tau, \quad t \geq a,
\]

then (2) has a solution \( s(t) \) on \([a, b)\) such that \( 0 < 1 - ts(t) \leq 1 - tr(t) \).

**Proof.** Choose \( s_0 \) such that \( ar(a) \leq as_0 \leq 1 \). Condition (3) now follows from (6) and the fact that \( 0 \leq 1 - as_0 \leq 1 - ar(a) \).

In order to obtain a result which complements Theorem 2 of [1], define \( z(t) = \mu'(t) - \mu(t) r(t) \) and \( v(t) = \mu'(t) - \mu(t) s(t) \) where \( \mu(t) \) is a positive function of class \( C^2[a, b) \). Then \( z(t) \) and \( v(t) \) satisfy

\[
\mu(t) z(t) = \mu(a) z(a) + \int_a^t \mu''(\tau) \mu(\tau) d\tau + \int_a^t z^2(\tau) d\tau + \int_a^t \mu^2(\tau) q_1(\tau) d\tau,
\]

\[
\mu(t) v(t) = \mu(a) v(a) + \int_a^t \mu''(\tau) \mu(\tau) d\tau + \int_a^t v^2(\tau) d\tau + \int_a^t \mu^2(\tau) q_2(\tau) d\tau.
\]

The following theorem can now be proven:

**Theorem 2.** Suppose that (1) has a solution \( r(t) \) on \([a, b)\) where \( 0 < a < b \leq \infty \), and that there exists a constant \( s_0 \) such that

\[
|\mu(a) - \mu(a) s_0| + \int_a^t \mu^2(\tau) q_2(\tau) d\tau \leq |\mu(a) - \mu(a) r(a)| + \int_a^t \mu^2(\tau) q_1(\tau) d\tau, \quad t \geq a,
\]

where \( \mu(t) \) is a positive function of class \( C^2[a, b) \).
where $\mu(t)$ is a positive function of class $C^2[a, b)$. Then (2) has a solution $s(t)$ on $[a, b)$ such that $s(a) = s_0$ and $|s'(t) - \mu(t) s(t)| \leq \mu'(t) - \mu(t)r(t)$. If in addition, we assume that $\mu(a)r(a) \leq \mu'(a)$, then condition (9) can be replaced by

$$
\left| \int_a^t \mu^2(r) q_2(r) \, dr \right| \leq \int_a^t \mu^2(r) q_1(r) \, dr.
$$

Theorem 2 can be used to obtain several new oscillation and disconjugacy criteria for the equation

(11) \hspace{1cm} y''(t) + q_1(t)y(t) = 0,

for example:

$$
\frac{1}{4} < \frac{1}{t-a} \int_a^t r^2 q_1(r) \, dr \rightarrow \text{oscillation}, \quad \left| \frac{1}{t-a} \int_a^t r^2 q_1(r) \, dr \right| < \frac{1}{4} \rightarrow \text{disconjugacy}.
$$

Theorem 2 can also be used to compare the disconjugacy of (11) with that of

(12) \hspace{1cm} y''(t) + q_2(t)y(t) = 0.

**Theorem 3.** If equation (11) is disconjugate on $[a, \infty)$ and

$$
\int_a^t (r-a) \alpha q_2(r) \, dr \leq \int_a^t (r-a) \alpha q_1(r) \, dr, \quad t \geq a,
$$

holds for some $\alpha \geq 2$, then equation (12) is disconjugate on $[a, \infty)$.

**Proof.** Let $Y(t)$ be the solution of (11) satisfying $Y(a) = 0, Y'(a) = 1$. Since equation (11) is disconjugate $Y(t) > 0$ for $t > a$ and $r(t) = Y'(t)/Y(t)$ is a solution of (1) existing on $(a, \infty)$. Condition (13) implies that $q_1(t) \geq 0$ in a positive neighborhood of $a$, from which it follows that

$$
\lim_{t \to a^+} \sup (t-a)r(t) \leq 1;
$$

(see the proof of Lemma 9.4.1 in [2, p. 457]). Now define $\mu(t) = (t-a)^{\alpha/2}$ and verify that for $\alpha \geq 2$ we have $\mu(a)r(a) \leq \mu'(a)$. The result now follows from Theorem 2.

In Example 3 of [1], Stafford and Heidel obtained the following sufficient condition for the disconjugacy of (11) on $[0, \infty)$:

$$
\frac{3}{4} \leq \frac{1}{t} \int_0^t r^2 q_1(r) \, dr \leq \frac{1}{4}.
$$

We will obtain a generalization of this condition with the aid of some observations made by A. Wintner [3] and the following lemma which we present without proof.

**Lemma 1.** Equation (11) is disconjugate on $[0, \infty)$ if and only if there exists a function $w(t)$ of class $C^1[0, \infty)$ satisfying the inequality

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Theorem 4. Suppose that there exists a nonnegative function $b(t)$ such that
\[
\int_0^t r^2[q_1(r) + b(r)] \, dr \leq t^2 b^{1/2}(t) \quad \text{for } t \in [0, \infty);
\]
then equation (11) is disconjugate on $[0, \infty)$.

Proof. Assuming the condition of the theorem, a direct substitution shows that (15) is satisfied by
\[
w(t) = \frac{1}{t} \int_0^t r^2 q_1(r) \, dr + \frac{1}{t} \int_0^t r^2 b(r) \, dr.
\]
Notice that condition (14) now follows from Theorem 4 with the choice $b(t) = 1/4 t^2$.

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REFERENCES

