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REMARKS ON A COMPARISON THEOREM FOR SCALAR RICCATI EQUATIONS

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ABSTRACT. A comparison theorem for scalar Riccati equations will be stated which contains a recent theorem of Stafford and Heidel.

In a recent paper [1], Stafford and Heidel obtain a comparison theorem for the scalar Riccati equations

$$r'(t) + r^2(t) + q_1(t) = 0, \tag{1}$$

$$s'(t) + s^2(t) + q_2(t) = 0. \tag{2}$$

Theorem (Stafford and Heidel). *Suppose that* q_1 *and* q_2 *are nonnegative on* $[a, b)$ *where* $0 < a < b \leq \infty$ *and that*

$$\int_a^t \tau^2 q_2(\tau)\, d\tau \leq \int_a^t \tau^2 q_1(\tau)\, d\tau, \qquad t \geq a.$$

If (1) *has a solution* $r(t)$ *on* $[a, b)$ *such that* $ar(a) < 1$ *then* (2) *has a solution on* $[a, b)$.

The following theorem is an improvement over the result of Stafford and Heidel.

Theorem 1. *Suppose that* (1) *has a solution* $r(t)$ *on* $[a, b)$ *where* $0 < a < b \leq \infty$, *and that there exists a constant* s_0 *such that*

$$\left| a(1 - as_0) + \int_a^t \tau^2 q_2(\tau)\, d\tau \right| \leq a(1 - ar(a)) + \int_a^t \tau^2 q_1(\tau)\, d\tau. \tag{3}$$

Then (2) *has a solution* $s(t)$ *on* $[a, b)$ *such that* $s(a) = s_0$ *and* $|1 - ts(t)| \leq 1 - tr(t)$.

Proof. The outline of the proof that we present here follows Hille [2, p. 462]. Consider the integral equations

$$tz(t) = az(a) + \int_a^t z^2(\tau)\, d\tau + \int_a^t \tau^2 q_1(\tau)\, d\tau, \tag{4}$$

$$tv(t) = a(1 - as_0) + \int_a^t v^2(\tau)\, d\tau + \int_a^t \tau^2 q_2(\tau)\, d\tau. \tag{5}$$

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It is easily verified that (4) has the solution $z(t) = 1 - tr(t)$ on $[a, b)$ which is the limit of a strictly increasing sequence $\{z_n(t)\}$ of successive approximations satisfying

$$tz_n(t) = az(a) + \int_a^t z_{n-1}^2(\tau)\,d\tau + \int_a^t \tau^2 q_1(\tau)\,d\tau,$$

$$z_0(t) = az(a) + \int_a^t \tau^2 q_1(\tau)\,d\tau.$$

If we establish that (5) has a solution $v(t)$ existing on $[a, b)$, then $s(t) = (1 - v(t))/t$ is a solution of (2) existing on $[a, b)$ satisfying $s(a) = s_0$. A solution of (5) can be constructed as the limit of the sequence $\{v_n(t)\}$ of solutions to

$$tv_n(t) = a(1 - as_0) + \int_a^t v_{n-1}^2(\tau)\,d\tau + \int_a^t \tau^2 q_2(\tau)\,d\tau,$$

$$v_0(t) = a(1 - as_0) + \int_a^t \tau^2 q_2(\tau)\,d\tau.$$

The convergence of the sequence $\{v_n(t)\}$ follows from the inequalities

$$|v_n(t)| \le z_n(t), \qquad |v_{n+1}(t) - v_n(t)| \le z_{n+1}(t) - z_n(t).$$

Corollary 1. *If* (1) *has a solution* $r(t)$ *on* $[a, b)$ *such that* $ar(a) \le 1$ *and*

$$\left|\int_a^t \tau^2 q_2(\tau)\,d\tau\right| \le \int_a^t \tau^2 q_1(\tau)\,d\tau, \qquad t \ge a, \tag{6}$$

then (2) *has a solution* $s(t)$ *on* $[a, b)$ *such that* $0 < 1 - ts(t) \le 1 - tr(t)$.

Proof. Choose s_0 such that $ar(a) \le as_0 \le 1$. Condition (3) now follows from (6) and the fact that $0 \le 1 - as_0 \le 1 - ar(a)$.

In order to obtain a result which complements Theorem 2 of [1], define $z(t) = \mu'(t) - \mu(t)r(t)$ and $v(t) = \mu'(t) - \mu(t)s(t)$ where $\mu(t)$ is a positive function of class $C^2[a, b)$. Then $z(t)$ and $v(t)$ satisfy

$$\mu(t)z(t) = \mu(a)z(a) + \int_a^t \mu''(\tau)\mu(\tau)\,d\tau + \int_a^t z^2(\tau)\,d\tau + \int_a^t \mu^2(\tau)q_1(\tau)\,d\tau, \tag{7}$$

$$\mu(t)v(t) = \mu(a)v(a) + \int_a^t \mu''(\tau)\mu(\tau)\,d\tau + \int_a^t v^2(\tau)\,d\tau + \int_a^t \mu^2(\tau)q_2(\tau)\,d\tau. \tag{8}$$

The following theorem can now be proven:

Theorem 2. *Suppose that* (1) *has a solution* $r(t)$ *on* $[a, b)$ *where* $0 < a < b \le \infty$, *and that there exists a constant* s_0 *such that*

$$\left|\mu(a)(\mu'(a) - \mu(a)s_0) + \int_a^t \mu^2(\tau)q_2(\tau)\,d\tau\right| \le \mu(a)(\mu'(a) - \mu(a)r(a)) + \int_a^t \mu^2(\tau)q_1(\tau)\,d\tau, \qquad t \ge a, \tag{9}$$

where $\mu(t)$ *is a positive function of class* $C^2[a, b)$. *Then* (2) *has a solution* $s(t)$ *on* $[a, b)$ *such that* $s(a) = s_0$ *and* $|\mu'(t) - \mu(t)s(t)| \leq \mu'(t) - \mu(t)r(t)$. *If in addition, we assume that* $\mu(a)r(a) \leq \mu'(a)$, *then condition* (9) *can be replaced by*

$$\left|\int_a^t \mu^2(\tau)q_2(\tau)\,d\tau\right| \leq \int_a^t \mu^2(\tau)q_1(\tau)\,d\tau. \tag{10}$$

Theorem 2 can be used to obtain several new oscillation and disconjugacy criteria for the equation

$$Y''(t) + q_1(t)Y(t) = 0, \tag{11}$$

for example:

$$\frac{1}{4} < \frac{1}{t-a}\int_a^t \tau^2 q_1(\tau)\,d\tau \longrightarrow \text{oscillation}, \qquad \left|\frac{1}{t-a}\int_a^t \tau^2 q_1(\tau)\,d\tau\right| < \frac{1}{4} \longrightarrow \text{disconjugacy}.$$

Theorem 2 can also be used to compare the disconjugacy of (11) with that of

$$Y''(t) + q_2(t)Y(t) = 0. \tag{12}$$

Theorem 3. *If equation* (11) *is disconjugate on* $[a, \infty)$ *and*

$$\left|\int_a^t (\tau - a)^\alpha q_2(\tau)\,d\tau\right| \leq \int_a^t (\tau - a)^\alpha q_1(\tau)\,d\tau, \qquad t \geq a, \tag{13}$$

holds for some $\alpha \geq 2$, *then equation* (12) *is disconjugate on* $[a, \infty)$.

Proof. Let $Y(t)$ be the solution of (11) satisfying $Y(a) = 0$, $Y'(a) = 1$. Since equation (11) is disconjugate $Y(t) > 0$ for $t > a$ and $r(t) = Y'(t)/Y(t)$ is a solution of (1) existing on (a, ∞). Condition (13) implies that $q_1(t) \geq 0$ in a positive neighborhood of a, from which it follows that

$$\limsup_{t \to a^+} (t - a)r(t) \leq 1;$$

(see the proof of Lemma 9.4.1 in [2, p. 457]). Now define $\mu(t) = (t-a)^{\alpha/2}$ and verify that for $\alpha \geq 2$ we have $\mu(a)r(a) \leq \mu'(a)$. The result now follows from Theorem 2.

In Example 3 of [1], Stafford and Heidel obtained the following sufficient condition for the disconjugacy of (11) on $[0, \infty)$:

$$-\frac{3}{4} \leq \frac{1}{t}\int_0^t \tau^2 q_1(\tau)\,d\tau \leq \frac{1}{4}. \tag{14}$$

We will obtain a generalization of this condition with the aid of some observations made by A. Wintner [3] and the following lemma which we present without proof.

Lemma 1. *Equation* (11) *is disconjugate on* $[0, \infty)$ *if and only if there exists a function* $w(t)$ *of class* $C^1[0, \infty)$ *satisfying the inequality*

(15) $$(tw(t))' \geq w^2(t) + t^2 q_1(t) \quad for\ t \in [0, \infty).$$

Theorem 4. *Suppose that there exists a nonnegative function* $b(t)$ *such that*

$$\left| \int_0^t \tau^2 [q_1(\tau) + b(\tau)]\, d\tau \right| \leq t^2 b^{1/2}(t) \quad for\ t \in [0, \infty);$$

then equation (11) *is disconjugate on* $[0, \infty)$.

Proof. Assuming the condition of the theorem, a direct substitution shows that (15) is satisfied by

$$w(t) = \frac{1}{t} \int_0^t \tau^2 q_1(\tau)\, d\tau + \frac{1}{t} \int_0^t \tau^2 b(\tau)\, d\tau.$$

Notice that condition (14) now follows from Theorem 4 with the choice $b(t) = 1/4t^2$.

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