A SUFFICIENT CONDITION FOR EVENTUAL DISCONJUGACY

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ABSTRACT. It is known that the scalar equation \( y^{(n)} + p_1(t)y^{(n-1)} + \cdots + p_n(t)y = 0, \ t > 0, \ n > 1, \) is eventually disconjugate if \( p_1, \ldots, p_n \in C[0, \infty) \) and \( \int_0^\infty |p_i(t)|^{i-1} \, dt < \infty, \ 1 \leq i \leq n. \) This paper presents a weaker integral condition which also implies that the given equation is eventually disconjugate.

A linear differential equation
\[
y^{(n)} + p_1(t)y^{(n-1)} + \cdots + p_n(t)y = 0, \quad t > 0, \quad (n > 1),
\]
is eventually disconjugate if there is an interval \([a, \infty)\) on which none of its nontrivial solutions has more than \(n-1\) zeros, counting multiplicities. From a theorem of Willett \([4, \text{Theorem 1.4}]\), (1) is eventually disconjugate if \( p_1, \ldots, p_n \in C[0, \infty) \) and
\[
\int_0^\infty |p_i(t)|^{i-1} \, dt < \infty, \quad 1 \leq i \leq n.
\]
This paper presents a weaker sufficient condition for eventual disconjugacy.

Let \( l \) be the set of functions defined for large \( t \) and integrable at \( \infty \), let \( A_0 \) be the set of functions in \( l \) and absolutely integrable at \( \infty \), and let \( \Phi \) be the set of functions which are positive, nondecreasing and absolutely continuous for large \( t \). From Abel's theorem \([1, \text{p. 476}]\), \( f/\Phi \in l \) if \( f \in l \) and \( \phi \in \Phi \), and
\[
\int_t^\infty f(s)(\phi(s))^{-1} \, ds = o(1/\phi(s)).
\]
(In this paper, '"O"' and '"o"' refer to behavior as \( t \to \infty.\))

For \( i \geq 1 \), define \( A_i \) as follows: \( f \in A_i \) if and only if \( f \in l \) and there are functions \( \phi_1, \ldots, \phi_i \) in \( \Phi \) such that if
\[
(4) \quad f_0 = f,
\]
\[
(5) \quad Q_j(t) = \int_t^\infty f_{j-1}(s)(\phi_j(s))^{-1} \, ds, \quad 1 \leq j \leq i,
\]
\[
(6) \quad f_j(t) = t^{-1} \phi_j(t)Q_j(t), \quad 1 \leq j \leq i,
\]
and
\[
(7) \quad g_j(t) = \phi'_j(t)Q_j(t), \quad 1 \leq j \leq i,
\]

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then \( f_0, \ldots, f_{i-1} \in \mathcal{I} \) and \( f_i, g_1, \ldots, g_i \in \mathcal{A}_0 \). Finally, define

\[(8) \quad B_{k-1} = \bigcup_{j=0}^{k-1} A_j, \quad k \geq 1.\]

Our main result is

**Theorem 1.** If \( p_1, \ldots, p_n \in C[0, \infty) \) and

\[(9) \quad p_{k-1}^k \in B_{k-1}, \quad 1 \leq k \leq n,\]

then (1) is eventually disconjugate.

To prove this theorem, we will show that its hypothesis implies that (1) has a fundamental set of solutions \( y_0', \ldots, y_{n-1} \) such that

\[(10) \quad y^{(j)}(t) = \begin{cases} \frac{t^{r-j}(1 + o(1))}{(r-j)!}, & 0 \leq j \leq r, \\ o(t^{r-j}), & r+1 \leq j \leq n-1. \end{cases}\]

If (10) holds, then the Wronskians \( W_r = W(y_0', \ldots, y_r) \) satisfy

\[(11) \quad W_r(t) = 1 + o(1), \quad 0 \leq r \leq n-1,\]

and are therefore positive on some interval \([a, \infty)\). Because of this, Pólya's disconjugacy condition [5] implies that (1) is disconjugate on \([a, \infty)\).

(To see that (10) implies (11), observe that a typical term in the expansion of \( W_r(t) \) according to the definition of determinant is of the form \( \pm \prod_{i=0}^{r-1} y^{(i)}(t) \), where \( \{j_0, \ldots, j_{r-1}\} \) is a permutation of \( \{0, \ldots, r-1\} \). The product for which \( j_i = i \) \( (0 \leq i \leq r-1) \) equals \( 1 + o(1) \) from (10). Every other product is of the order \( \prod_{i=0}^{r-1} O(t^{r-i}) \), where ""O"" can be replaced by ""o"" in at least one factor. Since \( \sum_{i=0}^{r-1} (i-j) = 0 \), every such product equals \( o(1) \).)

We will use the contraction mapping principle to show that \( y_0', \ldots, y_{n-1} \) exist. The subspace \( P_r[t_0, \infty) \subset C^{(r-1)}(t_0, \infty) \) consisting of functions such that \( y^{(j)}(t) = O(t^{r-j}) \), \( 0 \leq j \leq n-1 \), is a Banach space under the norm \( \sigma_r(t_0; y) \), where

\[
(12) \quad \sigma_r(t_0; y) = \sum_{j=0}^{n-1} \sup_{s \geq t} |s^{r-j}y^{(j)}(s)|.
\]

With

\[
(13) \quad (My)(t) = \sum_{k=1}^{n} p_k(t)y^{(n-k)}(t)
\]

(cf. (1)), define the mappings \( T_0, \ldots, T_{n-1} \) by

\[
(14) \quad (T_0y)(t) = 1 + \int_{t}^{\infty} \frac{(t-s)^{n-1}}{(n-1)!} (My)(s) \, ds.
\]
and

\[(T_r y)(t) = \frac{r^t}{r!} + \int_{t_0}^{t} \frac{(t - \lambda)^{r-1}}{(r-1)!} d\lambda \int_{\lambda}^{\infty} \frac{((\lambda - s)^{n-r-1} (My)(s) ds}{(n-r-1)!}, \quad r = 1, \ldots, n-1.\]

We will show that (9) implies \(T_r\) is a contraction mapping of \(P_r[t_0, \infty)\) into itself if \(t_0\) is sufficiently large. It will then follow from the contraction mapping principle [2, p. 11] that there is a function \(y_r\) in \(P_r[t_0, \infty)\) such that \(T_r y_r = y_r\). It is then straightforward to verify that \(y_r\) satisfies (1) and (10).

Thus, the proof of Theorem 1 is reduced to showing that \(T_r\) has the stated property. We do this by means of the following lemmas.

**Lemma 1.** Suppose \(f \in A, h \in C^{(i)}[t_0, \infty) (t_0 \geq 0), and \)

\[h^{(j)}(t) = O(t^{-j}), \quad 0 \leq j \leq i.\]

Then the integral

\[\int_{t}^{\infty} s^{-\alpha f(s)h(s)} ds, \quad \alpha > 0, \quad s \geq t_0, \]

exists. Moreover, there is a function \(m_i\), which depends on \(f\) but not on \(h\) or \(t_0\), such that

\[m_i(t) = o(1)\]

and

\[\int_{t}^{\infty} s^{-\alpha f(s)h(s)} ds \leq m_i(t) \sum_{j=0}^{i} \sup_{s \geq t} |s^j h^{(j)}(s)|, \quad 0 \leq \alpha \leq n-1, \quad t \geq t_0.\]

**Proof.** If \(i = 0\), (16) converges absolutely and (17) and (18) hold, with \(m_0(t) = \int_{t}^{\infty} |f(s)| ds\). If \(i \geq 1\), define

\[h_0 = h, \quad h_r = \frac{th'}{r} h'_{r-1} - \frac{a}{r-1} h_{r-1}, \quad 1 \leq r \leq i,\]

and observe that \(h_0, \ldots, h_i\) are bounded because of (15).

Repeated integration by parts yields

\[
\int_{t}^{t_1} s^{-\alpha f(s)h(s)} ds = - \sum_{j=1}^{i} Q_j(s) \phi_j(s) s^{-\alpha h_{j-1}(s)} \left|^{t_1}_{t}\right.
\]

\[+ \sum_{j=1}^{i} \int_{t}^{t_1} s^{-\alpha g_j(s)h_{j-1}(s) ds} \]

\[+ \int_{t}^{t_1} s^{-\alpha f_i(s)h_i(s) ds}, \quad t_1 \geq t \geq t_0,\]
where $\phi_j, f_j, g_j, Q_j$ ($1 \leq j \leq i$) are the functions introduced in defining the class $A_i$ (cf. (4)–(7)) and $h_0, \ldots, h_i$ are as defined in (19). Since

$$\lim_{t_1 \to \infty} Q_j(t_1)\phi_j(t_1) = 0, \ 1 \leq j \leq i,$$

(by Abel's lemma; cf. (3)), the boundedness of $h_0, \ldots, h_i$ and the absolute integrability of $f_i$ and $g_1, \ldots, g_i$ at $\infty$ imply that the right side of (20) converges as $t_1 \to \infty$. Hence, the integral (16) converges, and it is easy to verify from (20) that (17) and (18) hold, with

$$m_i(t) = K_i \left[ \int_t^\infty |f_i(s)| \, ds + \sum_{j=1}^i \left( |Q_j(t)|\phi_j(t) + \int_t^\infty |g_j(s)| \, ds \right) \right],$$

where $K_i$ is a constant, which does not depend on $h$, such that

$$\sup_{s \geq t} |h_i(s)| \leq K_i \sum_{j=0}^i \sup_{s \geq t} |s^j h_j(s)|, \quad 0 \leq r \leq i, \quad 0 \leq \alpha \leq n - 1.$$

The existence of $K_i$ follows from (15) and (19).

The next lemma follows immediately from (8) and Lemma 1.

**Lemma 2.** The integral (16) exists if $f \in B_{k-1}, x \in C^{(k-1)}[t_0, \infty)$, and

$$x^{(i)}(t) = O(t^{-j}), \quad 0 \leq j \leq k - 1.$$

Moreover, there is a function $\mu_{k-1}$, which depends on $f$, but not on $h$ or $t_0$, such that

$$\mu_{k-1}(t) = o(1)$$

and

$$\left| \int_t^\infty s^{-\alpha} f(s) h(s) \, ds \right| \leq \mu_{k-1}(t) \sum_{j=0}^{k-1} \sup_{s \geq t} |s^j h_j(s)|,$$

$$0 \leq \alpha \leq n - 1, \quad t \geq t_0.$$

**Lemma 3.** If the hypotheses of Theorem 1 are satisfied, then the function $z_r$ defined by

$$z_r(t) = \int_t^\infty \frac{(s-t)^{n-r-1}}{(n-r-1)!} (My)(s) \, ds, \quad 0 \leq r \leq n - 1,$$

(cf. (12)) is in $C^{(n-r)}[t_0, \infty)$ if $y \in P_r[t_0, \infty)$. Furthermore,

$$t^i |z_r^{(i)}(t)| \leq G_r(t) \sigma_r(t; y), \quad 0 \leq i \leq n - r - 1, \quad t \geq t_0,$$

where

$$G_r(t) = o(1)$$

and $G_r$ depends on the operator $M$, but not on $y$ or $t_0$.

**Proof.** Integrals of the form
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(26) \[ \int_{t}^{\infty} s^{q} p_{k}(s)y^{(n-k)}(s) \, ds, \quad 0 \leq q \leq n - r - 1, \]

which appear in (23), can be rewritten as

\[ \int_{t}^{\infty} s^{-n + q + r + 1} f_{k}(s) x_{k}(s) \, ds, \quad \text{with} \quad f_{k}(t) = t^{k-1} p_{k}(t) \]

and

(27) \[ x_{k}(t) = t^{n-k-1} y^{(n-k)}(t). \]

Since \( f_{k} \in B_{k-1} \) and \( x_{k} \) satisfies (21), Lemma 2 implies that (26) exists and, from (22),

(28) \[ \left| \int_{t}^{\infty} s^{-n-r-1} s^{q} p_{k}(s)y^{(n-k)}(s) \, ds \right| \leq \mu_{k-1}(t) \sum_{j=0}^{k-1} \sup_{s \geq t} |s^{j} x_{k}^{(j)}(s)|, \]

where \( \mu_{k-1} \) is as defined in Lemma 2, with \( f = f_{k} \). From (27), there is a constant \( \lambda_{rk} \) such that

(29) \[ \sum_{j=0}^{k-1} \sup_{s \geq t} |s^{j} x_{k}^{(j)}(s)| \leq \frac{\lambda_{rk}}{2} \sigma_{r}(t; y) \]

for every \( y \) in \( P_{r}[t_{0}, \infty) \), and now (24) and (25) follow from (23), (28), (29) and the inequality

\[ \frac{1}{(n - r - i - 1)!} \sum_{\nu=0}^{n-r-i-1} \binom{n-r-i-1}{\nu} \leq 2, \]

if we take

\[ G_{r}(t) = \sum_{k=1}^{n} \lambda_{rk} \mu_{k-1}(t). \]

Lemma 3 implies that \( T_{r} \), as defined by (13) and (14), maps \( P_{r}[t_{0}, \infty) \) into itself, for any \( t_{0} \geq 0 \). Moreover, if \( y \) and \( \tilde{y} \) are both in \( P_{r}[t_{0}, \infty) \), routine estimates based on (13), (14) and (24) yield

(30) \[ \sigma_{r}(t_{0}; T_{r}y - T_{r}\tilde{y} \leq nG_{r}(t_{0}) \sigma_{r}(t_{0}; y - \tilde{y}), \]

where

\[ \tilde{G}_{r}(t_{0}) = \sup_{t \geq t_{0}} G_{r}(t). \]

Because of (25), we can choose \( t_{0} \) so that \( \tilde{G}_{r}(t_{0}) < 1/n \), and then (30) implies that \( T_{r} \) is a contraction mapping of \( P_{r}[t_{0}, \infty) \) into itself.

The fixed point (function) \( y_{r} \) of \( T_{r} \) satisfies (1) on \( (t_{0}, \infty) \), and can be extended as a solution of (1) over \( (0, \infty) \). To see that \( y_{r} \) satisfies (10) for
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\[ r \leq j \leq n - 1, \text{ observe that} \]
\[ y_r^{(j)}(t) = \delta_{ri} + \int_t^\infty \frac{(t-s)^{n-j-1}}{(n-j-1)!} (My)_r(s) ds, \quad r \leq j \leq n - 1, \]

and apply Lemma 3, with \( y = y_r \). For \( 0 \leq j \leq r - 1 \),
\[ y_r^{(j)}(t) = \frac{t^{r-j}}{(r-j)!} + \int_0^t \frac{(t-s)^{r-j-1}}{(r-j-1)!} \frac{z_r(\lambda)}{d\lambda}, \]

with \( z_r \) as defined by (23), with \( y = y_r \). However,
\[ \left| t^{r-j} \int_0^t \frac{(t-s)^{r-j-1}}{(r-j-1)!} \frac{z_r(\lambda)}{d\lambda} \right| \leq t^{-1} \int_0^t |z_r(\lambda)| \, d\lambda, \]

which approaches zero as \( t \to \infty \). (This is obvious if the last integral on the right converges as \( t \to \infty \), and it follows from L'Hospital's rule and Lemma 3 if it diverges.) Thus, \( y_r \) satisfies (10) for \( 0 \leq j \leq r - 1 \), and the proof of Theorem 1 is complete.

To the author's knowledge, the classes \( A_1, A_2, \ldots \) have not been previously considered in the literature, and virtually all questions about them are open. For example, is \( A_{i-1} \subseteq A_i \)? How can one construct functions which are in \( A_i \), but not in \( B_{i-1} \)? Since we cannot answer these questions yet, we will for the present simply exhibit classes of functions which are in \( A_1 \) and \( A_2 \), but not in \( A_0 \). This will show that Theorem 1 implies eventual disconjugacy for some equations which do not satisfy (2).

**Theorem 2.** Suppose \( F \) is continuous and has a bounded antiderivative \( F_1 \) on \([0, \infty)\), and \( \psi \) is absolutely continuous and approaches zero monotonically as \( t \to \infty \). Then the function \( f = F\psi \) is in \( A_1 \) if
\[ \int_0^\infty t^{-1} |\psi(t)| \, dt < \infty. \]

**Proof.** Take \( \phi_1 = 1 \) in (5). Then
\[ Q_1(t) = \int_t^\infty F(s)\psi(s) \, ds = -F_1(t)\psi(t) - \int_t^\infty F_1(s)\psi'(s) \, ds, \]

and our hypotheses imply that \( Q_1(t) = O(\psi(t)) \), and therefore, from (6), \( f_1(t) = O(t^{-1}\psi(t)) \), so that (31) implies \( f_1 \in A_0 \). Since \( g_1 = 0 \), it follows that \( f \in A_1 \).

As an application of Theorem 2, consider the iterated logarithms:
\[ L_0(t) = t, \quad L_1(t) = \log t, \quad L_2(t) = \log \log t, \ldots, \quad L_i(t) = \log L_{i-1}(t). \]

Since \( L_i'(t) = [\prod_{j=0}^{i-1} L_j(t)]^{-1}, i \geq 1 \), the function
\[ \psi(t) = [L_i(t+\rho)]^{-a} \left[ \prod_{j=1}^{i-1} L_j(t+\rho) \right]^{-1}, \quad i \geq 1, \]
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(where \( p \) is a positive constant such that \( \psi \) is defined on \([0, \infty)\)) has the properties required in Theorem 2, provided \( \alpha > 1 \). Theorems 1 and 2 imply, for example, that the equation

\[
y^{(n)} + \frac{t^{-n+1}[\log(\log(t+2e))]^{-3/2}\sin t}{\log(t+2e)} y = 0
\]

is eventually disconjugate if \( n \geq 2 \). This equation does not satisfy (2).

**Theorem 3.** Suppose \( F \) is continuous on \([0, \infty)\) and has a bounded antiderivative \( F_1 \), which in turn has a bounded antiderivative \( F_2 \). Suppose also that \( \psi \) and \( \psi' \) both tend monotonically to zero as \( t \to \infty \), and \( \psi' \) is absolutely continuous. Then the function \( f = F \psi \) is in \( A_2 \).

**Proof.** From (5) (with \( \phi_1 = 1 \)),

\[
Q_1(t) = \int_t^\infty F(s)\psi(s) \, ds = -F_1(t)\psi(t) + F_2(t)\psi'(t) + \int_t^\infty F_2(s)\psi''(s) \, ds.
\]

Our hypotheses imply that the integral on the right equals \( O(\psi'(t)) \), so (6) and (33) yield

\[
f_1(t) = -F_1(t)t^{-1}\psi(t) + A(t)t^{-1}\psi'(t),
\]

where \( A \) is continuous and bounded on \([0, \infty)\). Letting \( \phi_2 = 1 \) in (5) yields

\[
Q_2(t) = -\int_t^\infty F_1(s)s^{-1}\psi(s) \, ds + \int_t^\infty A(s)s^{-1}\psi'(s) \, ds
\]

\[
= F_2(t)t^{-1}\psi(t) + \int_t^\infty F_2(s)(s^{-1}\psi(s))' \, ds + \int_t^\infty A(s)s^{-1}\psi'(s) \, ds.
\]

Our hypotheses and the boundedness of \( A \) imply that each term on the right is \( O(t^{-1}\psi(t)) \); hence (6) and (34) yield

\[
f_2(t) = O(t^{-2}\psi(t)) = o(t^{-2}),
\]

and therefore \( f_2 \in A_2 \). Since \( g_1 = g_2 = 0 \), it follows that \( f \in A_2 \).

Theorems 1 and 3 imply, for example, that the equation

\[
y^{(n)} + \frac{t^{-n+1}\sin t}{L_k(t+p)} y = 0
\]

is eventually disconjugate if \( n \geq 3 \) and \( L_k \) is any iterated logarithm (32) defined on \([p, \infty)\). This equation does not satisfy (2).

**References**


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