PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 52, October 1975

A SUFFICIENT CONDITION FOR EVENTUAL DISCONJUGACY

WILLIAM F. TRENCH

ABSTRACT. It is known that the scalar equation $y^{(n)} + p_1(t)y^{(n-1)} + \cdots + p_n(t)y = 0$, $t > 0$, $n > 1$, is eventually disconjugate if $p_1, \ldots, p_n \in C[0, \infty)$ and $\int^\infty |p_i(t)|t^{i-1}\,dt < \infty$, $1 \le i \le n$. This paper presents a weaker integral condition which also implies that the given equation is eventually disconjugate.

A linear differential equation

$$y^{(n)} + p_1(t)y^{(n-1)} + \cdots + p_n(t)y = 0, \qquad t > 0 \quad (n > 1), \tag{1}$$

is *eventually disconjugate* if there is an interval $[a, \infty)$ on which none of its nontrivial solutions has more than $n-1$ zeros, counting multiplicities. From a theorem of Willett [4, Theorem 1.4], (1) is eventually disconjugate if $p_1, \ldots, p_n \in C[0, \infty)$ and

$$\int^\infty |p_i(t)|t^{i-1}\,dt < \infty, \qquad 1 \le i \le n. \tag{2}$$

This paper presents a weaker sufficient condition for eventual disconjugacy.

Let I be the set of functions defined for large t and integrable at ∞, let A_0 be the set of functions in I and absolutely integrable at ∞, and let Φ be the set of functions which are positive, nondecreasing and absolutely continuous for large t. From Abel's theorem [1, p. 476], $f/\Phi \in I$ if $f \in I$ and $\phi \in \Phi$, and

$$\int_t^\infty f(s)(\phi(s))^{-1}\,ds = o(1/\phi(s)). \tag{3}$$

(In this paper, "O" and "o" refer to behavior as $t \to \infty$.)

For $i \ge 1$, define A_i as follows: $f \in A_i$ if and only if $f \in I$ and there are functions $\phi_1, \ldots, \phi_i$ in Φ such that if

$$f_0 = f, \tag{4}$$

$$Q_j(t) = \int_t^\infty f_{j-1}(s)(\phi_j(s))^{-1}\,ds, \quad 1 \le j \le i, \tag{5}$$

$$f_j(t) = t^{-1}\phi_j(t)Q_j(t), \qquad 1 \le j \le i, \tag{6}$$

and

$$g_j(t) = \phi_j'(t)\,Q_j(t), \qquad 1 \le j \le i, \tag{7}$$

Received by the editors June 7, 1974.
AMS (MOS) subject classifications (1970). Primary 34C10; Secondary 34D05.
Key words and phrases. Eventually disconjugate, linear.

then $f_0, \dots, f_{i-1} \in I$ and $f_i, g_1, \dots, g_i \in A_0$. Finally, define

(8) $$B_{k-1} = \bigcup_{j=0}^{k-1} A_j, \qquad k \geq 1.$$

Our main result is

Theorem 1. *If* $p_1, \dots, p_n \in C[0, \infty)$ *and*

(9) $$t^{k-1} p_k \in B_{k-1}, \qquad 1 \leq k \leq n,$$

then (1) *is eventually disconjugate.*

To prove this theorem, we will show that its hypothesis implies that (1) has a fundamental set of solutions $y_0, \dots, y_{n-1}$ such that

(10) $$y_r^{(j)}(t) = \begin{cases} t^{r-j}(1 + o(1))/(r-j)!, & 0 \leq j \leq r, \\ o(t^{r-j}), & r+1 \leq j \leq n-1. \end{cases}$$

If (10) holds, then the Wronskians $W_r = W(y_0, \dots, y_r)$ satisfy

(11) $$W_r(t) = 1 + o(1), \qquad 0 \leq r \leq n-1,$$

and are therefore positive on some interval $[a, \infty)$. Because of this, Pólya's disconjugacy condition [3] implies that (1) is disconjugate on $[a, \infty)$.

(To see that (10) implies (11), observe that a typical term in the expansion of $W_r(t)$ according to the definition of determinant is of the form $\pm\Pi_{i=0}^{r-1} y_i^{(j_i)}(t)$, where $\{j_0, \dots, j_{r-1}\}$ is a permutation of $\{0, \dots, r-1\}$. The product for which $j_i = i$ $(0 \leq i \leq r-1)$ equals $1 + o(1)$, from (10). Every other product is of the order $\Pi_{i=0}^{r-1} O(t^{i-j_i})$, where "$O$" can be replaced by "$o$" in at least one factor. Since $\Sigma_{i=0}^{r-1} (i - j_i) = 0$, every such product equals $o(1)$.)

We will use the contraction mapping principle to show that $y_0, \dots,$ y_{n-1} exist. The subspace $P_r[t_0, \infty)$ of $C^{(n-1)}[t_0, \infty)$ consisting of functions such that $y^{(j)}(t) = O(t^{r-j})$, $0 \leq j \leq n-1$, is a Banach space under the norm $\sigma_r(t_0; y)$, where

$$\sigma_r(t: y) = \sum_{j=0}^{n-1} \sup_{s \geq t} |s^{j-r} y^{(j)}(s)|.$$

With

(12) $$(My)(t) = \sum_{k=1}^{n} p_k(t) y^{(n-k)}(t)$$

(cf. (1)), define the mappings $T_0, \dots, T_{n-1}$ by

(13) $$(T_0 y)(t) = 1 + \int_t^\infty \frac{(t-s)^{n-1}}{(n-1)!} (My)(s)\, ds$$

and

$$(T_r y)(t) = \frac{t^r}{r!} + \int_{t_0}^{t} \frac{(t-\lambda)^{r-1}}{(r-1)!}\, d\lambda \int_{\lambda}^{\infty} \frac{(\lambda - s)^{n-r-1}}{(n-r-1)!} (My)(s)\, ds, \tag{14}$$
$$r = 1, \ldots, n-1.$$

We will show that (9) implies T_r is a contraction mapping of $P_r[t_0, \infty)$ into itself if t_0 is sufficiently large. It will then follow from the contraction mapping principle [2, p. 11] that there is a function y_r in $P_r[t_0, \infty)$ such that $T_r y_r = y_r$. It is then straightforward to verify that y_r satisfies (1) and (10).

Thus, the proof of Theorem 1 is reduced to showing that T_r has the stated property. We do this by means of the following lemmas.

Lemma 1. *Suppose* $f \in A_i$, $h \in C^{(i)}[t_0, \infty)$ $(t_0 \geq 0)$, *and*

$$h^{(j)}(t) = O(t^{-j}), \qquad 0 \leq j \leq i. \tag{15}$$

Then the integral

$$\int_t^{\infty} s^{-\alpha} f(s) h(s)\, ds, \qquad \alpha \geq 0, \quad t \geq t_0, \tag{16}$$

exists. Moreover, there is a function m_i, *which depends on* f *but not on* h *or* t_0, *such that*

$$m_i(t) = o(1) \tag{17}$$

and

$$\left| t^{\alpha} \int_t^{\infty} s^{-\alpha} f(s) h(s)\, ds \right| \leq m_i(t) \sum_{j=0}^{i} \sup_{s \geq t} |s^j h^{(j)}(s)|, \tag{18}$$
$$0 \leq \alpha \leq n-1, \quad t \geq t_0.$$

Proof. If $i = 0$, (16) converges absolutely and (17) and (18) hold, with $m_0(t) = \int_t^{\infty} |f(s)|\, ds$. If $i \geq 1$, define

$$h_0 = h, \quad h_r = t h'_{r-1} - \alpha h_{r-1}, \qquad 1 \leq r \leq i, \tag{19}$$

and observe that $h_0, \ldots, h_i$ are bounded because of (15).

Repeated integration by parts yields

$$\begin{aligned} \int_t^{t_1} s^{-\alpha} f(s) h(s)\, ds = &- \sum_{j=1}^{i} Q_j(s) \phi_j(s) s^{-\alpha} h_{j-1}(s) \Big|_t^{t_1} \\ &+ \sum_{j=1}^{i} \int_t^{t_1} s^{-\alpha} g_j(s) h_{j-1}(s)\, ds \\ &+ \int_t^{t_1} s^{-\alpha} f_i(s) h_i(s)\, ds, \qquad t_1 \geq t \geq t_0, \end{aligned} \tag{20}$$

where ϕ_j, f_j, g_j, Q_j $(1 \le j \le i)$ are the functions introduced in defining the class A_i (cf. (4)–(7)) and $h_0, \dots, h_i$ are as defined in (19). Since $\lim_{t_1 \to \infty} Q_j(t_1)\phi_j(t_1) = 0$, $1 \le j \le i$, (by Abel's lemma; cf. (3)), the boundedness of $h_0, \dots, h_i$ and the absolute integrability of f_i and $g_1, \dots, g_i$ at ∞ imply that the right side of (20) converges as $t_1 \to \infty$. Hence, the integral (16) converges, and it is easy to verify from (20) that (17) and (18) hold, with

$$m_i(t) = K_i \left[\int_t^\infty |f_i(s)|\, ds + \sum_{j=1}^{i} \left(|Q_j(t)|\phi_j(t) + \int_t^\infty |g_j(s)|\, ds \right) \right],$$

where K_i is a constant, which does not depend on h, such that

$$\sup_{s \ge t} |h_r(s)| \le K_i \sum_{j=0}^{i} \sup_{s \ge t} |s^j h^{(j)}(s)|, \qquad 0 \le r \le i, \qquad 0 \le \alpha \le n-1.$$

The existence of K_i follows from (15) and (19).

The next lemma follows immediately from (8) and Lemma 1.

Lemma 2. *The integral* (16) *exists if* $f \in B_{k-1}$, $x \in C^{(k-1)}[t_0, \infty)$, *and*

$$x^{(j)}(t) = O(t^{-j}), \quad 0 \le j \le k-1. \tag{21}$$

Moreover, there is a function μ_{k-1}, *which depends on* f, *but not on* h *or* t_0, *such that*

$$\mu_{k-1}(t) = o(1)$$

and

$$\left| t^\alpha \int_t^\infty s^{-\alpha} f(s) h(s)\, ds \right| \le \mu_{k-1}(t) \sum_{j=0}^{k-1} \sup_{s \ge t} |s^j h^{(j)}(s)|, \qquad 0 \le \alpha \le n-1, \quad t \ge t_0. \tag{22}$$

Lemma 3. *If the hypotheses of Theorem* 1 *are satisfied, then the function* z_r *defined by*

$$z_r(t) = \int_t^\infty \frac{(t-s)^{n-r-1}}{(n-r-1)!} (My)(s)\, ds, \qquad 0 \le r \le n-1, \tag{23}$$

(cf. (12)) *is in* $C^{(n-r)}[t_0, \infty)$ *if* $y \in P_r[t_0, \infty)$. *Furthermore,*

$$t^i |z_r^{(i)}(t)| \le G_r(t)\sigma_r(t; y), \qquad 0 \le i \le n-r-1, \quad t \ge t_0, \tag{24}$$

where

$$G_r(t) = o(1) \tag{25}$$

and G_r *depends on the operator* M, *but not on* y *or* t_0.

Proof. Integrals of the form

$$(26)\qquad \int_t^\infty s^q p_k(s) y^{(n-k)}(s)\,ds, \qquad 0 \le q \le n-r-1,$$

which appear in (23), can be rewritten as

$$\int_t^\infty s^{-n+q+r+1} f_k(s) x_k(s)\,ds, \quad \text{with } f_k(t) = t^{k-1} p_k(t)$$

and

$$(27)\qquad x_k(t) = t^{n-k-r} y^{(n-k)}(t).$$

Since $f_k \in B_{k-1}$ and x_k satisfies (21), Lemma 2 implies that (26) exists and, from (22),

$$(28)\qquad \left| t^{n-q-r-1} \int_t^\infty s^q p_k(s) y^{(n-k)}(s)\,ds \right| \le \mu_{k-1}(t) \sum_{j=0}^{k-1} \sup_{s \ge t} |s^j x_k^{(j)}(s)|,$$

where μ_{k-1} is as defined in Lemma 2, with $f = f_k$. From (27), there is a constant λ_{rk} such that

$$(29)\qquad \sum_{j=0}^{k-1} \sup_{s \ge t} |s^j x_k^{(j)}(s)| \le \frac{\lambda_{rk}}{2} \sigma_r(t; y)$$

for every y in $P_r[t_0, \infty)$, and now (24) and (25) follow from (23), (28), (29) and the inequality

$$\frac{1}{(n-r-i-1)!} \sum_{\nu=0}^{n-r-i-1} \binom{n-r-i-1}{\nu} = \frac{2^{n-r-i-1}}{(n-r-i-1)!} \le 2,$$

if we take

$$G_r(t) = \sum_{k=1}^{n} \lambda_{rk} \mu_{k-1}(t).$$

Lemma 3 implies that T_r, as defined by (13) and (14), maps $P_r[t_0, \infty)$ into itself, for any $t_0 \ge 0$. Moreover, if y and $\tilde{y}$ are both in $P_r[t_0, \infty)$, routine estimates based on (13), (14) and (24) yield

$$(30)\qquad \sigma_r(t_0; T_r y - T_r \tilde{y} \le n \tilde{G}_r(t_0) \sigma_r(t_0; y - \tilde{y}),$$

where

$$\tilde{G}_r(t_0) = \sup_{t \ge t_0} G_r(t).$$

Because of (25), we can choose t_0 so that $\tilde{G}_r(t_0) < 1/n$, and then (30) implies that T_r is a contraction mapping of $P_r[t_0, \infty)$ into itself.

The fixed point (function) y_r of T_r satisfies (1) on (t_0, ∞), and can be extended as a solution of (1) over $(0, \infty)$. To see that y_r satisfies (10) for

$r \leq j \leq n-1$, observe that

$$y_r^{(j)}(t) = \delta_{rj} + \int_t^\infty \frac{(t-s)^{n-j-1}}{(n-j-1)!}(My_r)(s)\,ds, \qquad r \leq j \leq n-1,$$

and apply Lemma 3, with $y = y_r$. For $0 \leq j \leq r-1$,

$$y_r^{(j)}(t) = \frac{t^{r-j}}{(r-j)!} + \int_{t_0}^t \frac{(t-\lambda)^{r-j-1}}{(r-j-1)!}\, z_r(\lambda)\,d\lambda,$$

with z_r as defined by (23), with $y = y_r$. However,

$$\left| t^{j-r} \int_{t_0}^t (t-\lambda)^{r-j-1} y_r(\lambda)\,d\lambda \right| \leq t^{-1} \int_{t_0}^t |z_r(\lambda)|\,d\lambda,$$

which approaches zero as $t \to \infty$. (This is obvious if the last integral on the right converges as $t \to \infty$, and it follows from L'Hospital's rule and Lemma 3 if it diverges.) Thus, y_r satisfies (10) for $0 \leq j \leq r-1$, and the proof of Theorem 1 is complete.

To the author's knowledge, the classes $A_1, A_2, \cdots$ have not been previously considered in the literature, and virtually all questions about them are open. For example, is $A_{i-1} \subset A_i$? How can one construct functions which are in A_i, but not in B_{i-1}? Since we cannot answer these questions yet, we will for the present simply exhibit classes of functions which are in A_1 and A_2, but not in A_0. This will show that Theorem 1 implies eventual disconjugacy for some equations which do not satisfy (2).

Theorem 2. *Suppose F is continuous and has a bounded antiderivative F_1 on $[0, \infty)$, and ψ is absolutely continuous and approaches zero monotonically as $t \to \infty$. Then the function $f = F\psi$ is in A_1 if*

(31) $$\int^\infty t^{-1} |\psi(t)|\,dt < \infty.$$

Proof. Take $\phi_1 = 1$ in (5). Then

$$Q_1(t) = \int_t^\infty F(s)\psi(s)\,ds = -F_1(t)\psi(t) - \int_t^\infty F_1(s)\psi'(s)\,ds,$$

and our hypotheses imply that $Q_1(t) = O(\psi(t))$, and therefore, from (6), $f_1(t) = O(t^{-1}\psi(t))$, so that (31) implies $f_1 \in A_0$. Since $g_1 = 0$, it follows that $f \in A_1$.

As an application of Theorem 2, consider the iterated logarithms:

(32) $L_0(t) = t,\ L_1(t) = \log t,\ L_2(t) = \log(\log t), \ldots,\ L_i(t) = \log L_{i-1}(t).$

Since $L_i'(t) = [\prod_{j=0}^{i-1} L_j(t)]^{-1}$, $i \geq 1$, the function

$$\psi(t) = [L_i(t+\rho)]^{-\alpha} \left[\prod_{j=1}^{i-1} L_j(t+\rho) \right]^{-1}, \qquad i \geq 1,$$

(where ρ is a positive constant such that ψ is defined on $[0, \infty)$) has the properties required in Theorem 2, provided $\alpha > 1$. Theorems 1 and 2 imply, for example, that the equation

$$y^{(n)} + \frac{t^{-n+1}[\log(\log(t+2e))]^{-3/2}\sin t}{\log(t+2e)}\, y = 0$$

is eventually disconjugate if $n \geq 2$. This equation does not satisfy (2).

Theorem 3. *Suppose* F *is continuous on* $[0, \infty)$ *and has a bounded antiderivative* F_1, *which in turn has a bounded antiderivative* F_2. *Suppose also that* ψ *and* ψ' *both tend monotonically to zero as* $t \to \infty$, *and* ψ' *is absolutely continuous. Then the function* $f = F\psi$ *is in* A_2.

Proof. From (5) (with $\phi_1 = 1$),

$$Q_1(t) = \int_t^\infty F(s)\psi(s)\,ds = -F_1(t)\psi(t) + F_2(t)\psi'(t) + \int_t^\infty F_2(s)\psi''(s)\,ds. \tag{33}$$

Our hypotheses imply that the integral on the right equals $O(\psi'(t))$, so (6) and (33) yield

$$f_1(t) = -F_1(t)t^{-1}\psi(t) + A(t)t^{-1}\psi'(t),$$

where A is continuous and bounded on $[0, \infty)$. Letting $\phi_2 = 1$ in (5) yields

$$\begin{aligned} Q_2(t) &= -\int_t^\infty F_1(s)s^{-1}\psi(s)\,ds + \int_t^\infty A(s)s^{-1}\psi'(s)\,ds \\ &= F_2(t)t^{-1}\psi(t) + \int_t^\infty F_2(s)(s^{-1}\psi(s))'\,ds + \int_t^\infty A(s)s^{-1}\psi'(s)\,ds. \end{aligned} \tag{34}$$

Our hypotheses and the boundedness of A imply that each term on the right is $O(t^{-1}\psi(t))$; hence (6) and (34) yield

$$f_2(t) = O(t^{-2}\psi(t)) = o(t^{-2}),$$

and therefore $f_2 \in A_0$. Since $g_1 = g_2 = 0$, it follows that $f \in A_2$.

Theorems 1 and 3 imply, for example, that the equation

$$y^{(n)} + \frac{t^{-n+1}\sin t}{L_k(t+\rho)}\, y = 0$$

is eventually disconjugate if $n \geq 3$ and L_k is any iterated logarithm (32) defined on $[\rho, \infty)$. This equation does not satisfy (2).

REFERENCES

1. T. J. I'a. Bromwich, *An introduction to the theory of infinite series*, Macmillan, New York and London, 1955.

2. W. A. Coppel, *Stability and asymptotic behavior of differential equations*, Heath, Boston, Mass., 1955.

3. G. Pólya, *On the mean-value theorem corresponding to a given linear homogeneous differential equation*, Trans. Amer. Math. Soc. 24 (1924), 312–324.

4. D. Willett, *Disconjugacy tests for singular linear differential equations*, SIAM. J. Math. Anal. 2 (1971), 536–545. MR 46 #3904.

DEPARTMENT OF MATHEMATICS, DREXEL UNIVERSITY, PHILADELPHIA, PENNSYLVANIA 19104