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ASYMPTOTIC BEHAVIOR AND NONOSCILLATION OF VOLTERRA INTEGRAL EQUATIONS AND FUNCTIONAL DIFFERENTIAL EQUATIONS[1]

A. F. IZÉ AND A. A. FREIRIA

ABSTRACT. It is proved that if $q_{ij}(t, s)\rho_j(s)[\rho_i(t)]^{-1}$ is bounded, $i, j = 1, 2, \ldots, n$, and $f(t, x, x(u(s)))$ is "small",

$$x(u(s)) = (x_1(u_1(s)), x_2(u_2(s)), \ldots, x_n(u_n(s)))$$

with $u_i(t) \leq t$ and $\lim_{t\to\infty} u_i(t) = \infty$, the solutions of the integral equation

$$x(t) = h(t) + \int_0^t q(t, s)f(s, x(s), x(u(s)))\, ds$$

satisfy the conditions $x(t) = h(t) + \rho(t)a(t)$, $\lim_{t\to\infty} a(t) =$ constant where $\rho(t)$ is a nonsingular diagonal matrix chosen in such a way that $\rho^{-1}(t)h(t)$ is bounded. The results contain, in particular, some results on the asymptotic behavior, stability and existence of nonoscillatory solutions of functional differential equations.

Consider the system of Volterra integral equations

$$x(t) = h(t) + \int_0^t q(t, s)f(s, x(s), x(u(s)))\, ds + \int_0^t q(t, s)g(s)\, ds \tag{1}$$

where $h = (h_1, h_2, \ldots, h_n)$, $f = (f_1, f_2, \ldots, f_n)$, $g = (g_1, g_2, \ldots, g_n)$ are column vectors in E^n, $q = (q_{ij})$ is a $n \times n$ matrix in Euclidean n-dimensional space, $x(t) = (x_1(t), \ldots, x_n(t))$, $x(u(t)) = (x_1(u_1(t)), x_2(u_2(t)), \ldots, x_n(u_n(t)))$, $u_i(t)$ continuous, $u_i(t) \leq t$, $t \geq 0$, $\lim_{t\to\infty} u_i(t) = \infty$, $i = 1, 2, \ldots, n$.

We show here that the results obtained in [2] are also true for equation (1) which is more realistic physically; furthermore some mathematical models arising in biology are described by equations of type (1) and, in particular, by retarded differential equations.

We assume the hypotheses:

(H_1) $|q(t)|$ and $|g(t)| \in L(0, c)$, $0 < c < \infty$,

(H_2) $h(t)$ is continuous for $0 \leq t < \infty$,

(H_3) $f \in C([0, \infty) \times E^n \times E^n, E^n)$, $E^n = R^n$ or C^n.

These conditions guarantee the local existence of continuous solutions and continuability of each solution so long as they remain bounded; the proofs,

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as well as the proofs of Lemma 2 in [2], are the same, with small changes, given in [7] or [8].

We assume the hypotheses:

$$|f_i(t, x, y)| \leq \sum_{i=1}^{n} \epsilon_{ij}(t)|x_j|^{p_j} + \sum_{j=1}^{n} \epsilon'_{ij}(t)|y_j|^{q_j},$$

$$\epsilon_{ij} \geq 0, \quad \epsilon'_{ij} \geq 0, \quad p_j, q_j > 0, \quad i, j = 1, 2, \dots, n,$$

$$(\mathrm{H}_4) \qquad |x| < H, \quad |y| < H, \quad H \leq \infty,$$

$$\int^{\infty} \epsilon_{ij}(t)|\rho_j(t)|^{p_j}|\rho_i(t)|^{-1}\, dt < \infty, \qquad |\rho_i(t)| > 0, \quad i, j = 1, 2, \dots, n,$$

$$\int^{\infty} \epsilon'_{ij}(t)|\rho_j(u_j)|^{q_j}|\rho_i(t)|^{-1}\, dt < \infty,$$

(H_5) $\int^{\infty} |g_i(t)|\, |\rho_i(t)|^{-1}\, dt < \infty$,

(H_6) $q_{ij}(t, s)\rho_j(s)[\rho_i(t)]^{-1}$ are bounded for $0 \leq s \leq t < \infty$, $i, j = 1, 2, \dots, n$.

(H'_6) (H_6) is satisfied and $\lim_{t\to\infty} q_{ij}(t, s)[\rho_i(t)]^{-1}$ exists for almost all values of s.

Theorem 1. *Assume, with respect to equation* (1), (H_4), (H_5) *and* (H_6) *and let r be the maximum of the p_i, q_i, $i = 1, 2, \dots, n$. Then*

(A) *If $r > 1$ there exists $c > 0$ depending on r and ϵ_{ij}, ϵ'_{ij}, $i = 1, 2, \dots, n$, such that $|h_i(t)[\rho_i(t)]^{-1}| \leq c$, $i = 1, 2, \dots, n$, implies that the solutions of* (1) *exist in* $[0, \infty)$ *and satisfy the condition $x_i(t) = h_i(t) + a_i(t)\rho_i(t)$ where $a_i(t)$ is bounded. If, instead of condition* (H_6), (1) *satisfies* (H'_6) *then $\lim_{t\to\infty} a_i(t) = a_i$. Furthermore if $\lim_{t\to\infty} h_i(t)[\rho_i(t)]^{-1} = h_i$ constant, then $\lim_{t\to\infty}(x_i(t)/\rho_i(t)) = b_i$ constant.*

(B) *If $r \leq 1$, c can be chosen arbitrarily.*

Proof.

$$\frac{|x_i(t)|}{|\rho_i(t)|} \leq K_i + \int_0^t \sum_{r=1}^{n} \frac{|q_{ir}(t, s)|\, |\rho_r(s)|}{|\rho_i(t)|} \sum_{j=1}^{n} \epsilon_{rj}(s)|\rho_j(s)|^{p_j}|\rho_r^{-1}(s)| \frac{|x_j(s)|^{p_j}}{|\rho_j(s)|^{p_j}}\, ds$$

$$+ \int_0^t \sum_{r=1}^{n} \frac{|q_{ir}(t, s)|\, |\rho_r(s)|}{|\rho_i(t)|} \sum_{j=1}^{n} \epsilon'_{rj}(s)|\rho_j(u_j(s))|^{q_j}|\rho_r^{-1}(s)| \frac{|x_j(s)|^{q_j}}{|\rho_j(u_j(s))|^{q_j}}\, ds$$

$$+ \int_0^t \sum_{j=1}^{n} \frac{|q_{ij}(t, s)|\, |\rho_j(s)\rho_j(s)|^{-1}}{|\rho_i(t)|} |g_j(s)|\, ds$$

$$\le K_i + \int_0^t \sum_{r=1}^n C_r \sum_{j=1}^n \epsilon_{rj}(s)|\rho_j(s)|^{p_j}\rho_r(s)^{-1} \frac{|x_j(s)|^{p_j}}{|\rho_j(s)|^{p_j}}\, ds$$

$$+ \int_0^t \sum_{r=1}^n C_r \sum_{j=1}^n \epsilon'_{rj}(s)|\rho_j(u_j(s))|^{q_j}|\rho_r(s)|^{-1} \frac{|x_j(s)|^{q_j}}{|\rho_j(u_j(s))|^{q_j}}\, ds$$

$$+ \int_0^t \sum_{r=1}^n C_j |g_j(s)\rho_j(s)^{-1}|\, ds$$

$$\le K_i + K\int_0^t \left(\sum_{r,j=1}^n \epsilon_{rj}(s)|\rho_j(s)|^{p_j}\rho_r(s)^{-1} \sum_{i=1}^n \frac{|x_i(s)|^{p_i}}{|\rho_i(s)|^{p_i}}\right) ds$$

$$+ K\int_0^t \left(\sum_{r,j=1}^n \epsilon'_{rj}(s)[\rho_j(u_j(s))]^{q_j}\rho_r(s)^{-1} \sum_{i=1}^n \frac{|x_i(s)|^{q_i}}{|\rho_i(u_i(s))|^{q_i}}\right) ds$$

$$= K_i + K\int_0^t a(s)\left(\sum_{k=1}^n \frac{|x_k(s)|^{p_k}}{|\rho_k(s)|^{p_k}}\right) ds$$

$$+ K\int_0^t b(s)\left(\sum_{k=1}^n \frac{|x_k(s)|^{q_k}}{|\rho_k(u_k)|^{q_k}}\right) ds$$

$$= F_i(t).$$

Since $F_i(t)$ is monotone nondecreasing and $u_i(t) \le t$ for every i we have

$$|x_i(u_i(t))|/|\rho_i(u_i(t))| \le F_i(u_i(t)) \le F_i(t), \qquad i = 1, 2, \dots, n,$$

but

$$F'_i(t) = Ka(t)\left(\sum_{k=1}^n \frac{|x_k(t)|^{p_k}}{|\rho_k(t)|^{p_k}}\right) + Kb(t)\left(\sum_{k=1}^n \frac{|x_k(u_k(t))|^{q_k}}{|\rho_k(u_k(t))|^{q_k}}\right)$$

$$\le Ka(t)\sum_{k=1}^n F_k(t)^{p_k} + Kb(t)\sum_{k=1}^n F_k(t)^{q_k}.$$

If for every k, $F_k(t) \le 1$, $|x_k(t)|/|\rho_k(t)| \le F_k(t)$ are bounded, and that is what we ultimately want to prove, if for some k, $F_k(t) > 1$ by [2, Lemma 3],

$$\sum_{k=1}^n F_k(t)^{p_k} \le nK\left(\sum_{k=1}^n F_k(t)\right)^p, \qquad \sum_{k=1}^n F_k(t)^{q_k} \le n\left(\sum_{k=1}^n F_k(t)\right)^q,$$

$p = \max_k p_k$, $q = \max_k q_k$. If $r = \max(p, q)$ then

$$F_i'(t) \leq nK(a(t) + b(t))F(t)^r$$

where $F(t) = \Sigma_{k=1}^n F_k(t)$ and $F'(t) \leq n^2K(a(t) + b(t))F(t)^r$.

$\Sigma_{i=1}^n (|x_i(t)|/|\rho_i(t)|) \leq F(t) \leq z(t)$ where $z(t)$ is the maximal solution of the differential equation

$$\dot{z} = n^2K(a(t) + b(t))z(t)^r \tag{2}$$

with the same reasoning made in [2], it is easy to show that the solutions of equation (2) are bounded when $r > 1$ if $|z(0)| \leq C$ for C small enough and are globally bounded if $r \leq 1$.

Thus in the conditions of hypothesis (H_6), $x_i(t) = h_i(t) + \rho_i(t)a_i(t)$ where

$$a_i(t) = \int_0^t \sum_{r=1}^n \frac{q_{ir}(t, s)}{\rho_i(t)} f_r(s, x(s), x(u(s)))\, ds + \int_0^t \sum_{j=1}^n \frac{q_{ij}(t, s)}{\rho_i(t)} g_r(s)\, ds$$

is bounded and in the conditions of hypothesis (H_6') by [2, Lemma 1], $\lim_{t\to\infty} a_i(t) = \text{constant}$.

Remark 1. The results above are also true for the more general integral equation

$$x(t) = h(t) + \int_0^t F(t, s, x(s), x(u(s)))\, ds \tag{3}$$

where F is continuous in t and x and $|F|$ is locally Lebesgue integrable and satisfies the hypothesis

$$\begin{aligned} &|F_i(t, s, x(s), x(u(s)))|\, |\rho_i(t)|^{-1} \\ (H_7) \qquad &\leq \sum_{j=1}^n g_{ij}(s)|x_j|^{p_j} + \sum_{j=1}^n g_{ij}'(s)|x_j(u_j(s))|^{q_j}, \end{aligned}$$

$|x| < H < \infty$ where $g_{ij} \geq 0$, $g_{ij}' > 0$, $p_j \geq 0$, $q_j > 0$, $i, j = 1, 2, \ldots, n$, and

$$\int^\infty g_{ij}(t)|\rho_j(t)|^{p_j}\, dt < \infty, \quad \int^\infty g_{ij}(t)|\rho_j(u_j(t))|^{q_j}\, dt < \infty$$

or

(H_7') (H_7) is satisfied and $\lim_{t\to\infty} F_i(t, s, x(s), x(u(s)))[\rho_i(t)]^{-1}$ exists and is finite for almost all values of s.

Then conditions (A) and (B) of Theorem 1 are satisfied for equation (3). Still more generally let $\rho = (\rho_i(t))$ be a diagonal matrix, $|\rho_i(t)| > 0$, $i = 1, 2, \ldots, n$, chosen in such a way that $|\rho(t)^{-1}h(t)| < K$. Suppose that

$$|\rho(t)^{-1}F(t, s, x(s), x(u(s)))| < \omega(t, s, |\rho(s)^{-1}x(s)|, |\rho(u(s))^{-1}x(u(s))|)$$

where $\omega(t, s, r_1, r_2)$ is monotone in r_j for $i \neq j$ for each (t, s, r_i). Then if the solutions of the integral equation

$$y(t) = K + \int_0^t \omega(t, s, y(s), y(u(s)))\, ds$$

are bounded, the solutions of (3) satisfy $x(t) = h(t) + \rho(t)a(t)$ with $a(t)$ bounded. If in addition

$$\lim_{t\to\infty} \rho(t)^{-1}F(t,\ s,\ x(s),\ x(u(s)))$$

exists and is finite for almost all values of s then $\lim_{t\to\infty} d(t) = a$ constant.

Remark 2. Theorem 1 is quite general to include several results on the asymptotic behavior and stability of Volterra integral equations and functional differential equations; for example, if $\rho_i(t) = k \cdot e^{-\alpha t}$, $\alpha \geq 0$, for every i, $|h(t)| < k \cdot e^{-\alpha t}$ and if $|q_{ij}(t, s)| < k \cdot e^{-\alpha(t-s)}$ then (H_6) is satisfied and if (H_4) is satisfied with $\int^\infty \epsilon_{ij}(t) \cdot e^{-\alpha(q_j-1)t}\, dt < \infty$ and $\int^\infty \epsilon'_{ij}(t)^{-\alpha(q_j-1)t}\, dt < \infty$ then the solutions of (1) in a neighborhood of the origin are uniformly asymptotically stable if $\alpha > 0$ and uniformly stable if $\alpha = 0$; in particular, the representation of the solution of the functional differential equation

$$\dot{x} = L(t,\ x_t) + f(t,\ x(t),\ x(u(t))), \qquad x_t(\theta) = x(t+\theta),$$
$$\Omega \subset R \times C([-r,\ 0],\ E^n), \qquad L: \Omega \longrightarrow E^n, \tag{4}$$

is linear, is given by

$$x(t) = T(t)\phi + \int_0^t U(t,\ s)f(s,\ x(s),\ x(u(s)))\, ds, \qquad T: C \longrightarrow C,$$

and if the solutions of

$$\dot{y} = L(t,\ y_t) \tag{5}$$

are uniformly asymptotically stable, then $\|T(t)\| \leq k \cdot e^{-\alpha t}$, $\alpha > 0$, and $|U(t,\ s)| \leq k \cdot e^{-\alpha(t-s)}$ and, by the reasoning above, if f satisfies (H_6) then the solutions of (4) with $x(0)$ small enough are uniformly asymptotically stable; in particular, if the solutions of (5) are uniformly stable, $\|T(t)\phi\|$ is bounded and $|U(t,\ s)|$ is bounded, and if f satisfies (H_6) with $\rho_i(t) = k$ constant, then the solutions of (4) are uniformly stable if $x(0)$ is small enough.

Consider now the systems

$$\dot{x} = A(t)x + f(t,\ x(t),\ x(u(t))) + g(t), \tag{6}$$

$$\dot{} = A(t)y \tag{7}$$

satisfying the conditions:

$$|f_i(t,\ x,\ y)| \leq \sum_{i=1}^n \epsilon_{ij}(t)|x_j|^{p_j} + \sum_{j=1}^n \epsilon'_{ij}(t)|y_j|^{q_j},$$

$$\epsilon_{ij} \geq 0, \quad \epsilon'_{ij} \geq 0, \qquad p_j,\ q_j > 0, \qquad i,\ j = 1,\ 2,\ \ldots,\ n,$$

$$|x| < H, \quad |y| < H, \quad H \leq \infty,$$

(H'_4)
$$\int^\infty \epsilon_{ij}(t)|\det \rho(t)|\ \exp\left[-\int_{T_0}^t T_r A(s)\, ds\right] |\rho_j(t)|^{p_j}|\rho_i(t)|^{-1}\, dt < \infty,$$

$$i,\ j = 1,\ 2,\ \ldots,\ n, \qquad |\rho_i(t)| > 0,$$

$$\int^\infty \epsilon'_{ij}(t)|\det \rho(t)|\ \exp\left[-\int_{T_0}^t T_r A(s)\, ds\right] |\rho_j(u_j)|^{q_j}|\rho_i(t)|^{-1}\, dt < \infty,$$

$$(\mathrm{H}_5') \qquad \int^{\infty} |g_i(t)|\,|\det \rho(t)|\,\exp\left[-\int_{T_0}^{t} T_r A(s)\,ds\right] |\rho_i(t)|^{-1}\,dt < \infty.$$

The following theorem generalizes Theorem 1 of Ladas [5], Theorem 1 of Marušiak [6] for delay equations and several theorems on the asymptotic behavior of ordinary differential equations, in particular Theorem 2.1 of Hallam [2], Theorem 1 of Waltman [9], Theorem 2.1 of Izé [3], and was proved in [4] when (7) is the particular system $\dot{y}_i = C_j t^{\alpha_i - \alpha_j - 1} y_j$ considered by Izé in [3].

Theorem 2. *Suppose there exists a diagonal matrix $\rho(t) = (\rho_i(t))$, $|\rho_i(t)| > 0$, $i = 1, 2, \ldots, n$, such that for every solution $y(t)$ of (7), $y_i(t) = (a_i + o(1))\rho_i(t)$. Let f and g satisfy (H_4') and (H_5') and let $r = \max_i(p_i, q_i)$. Then: (A) If $r > 1$, for every solution $y(t) = (y_1, y_2, \ldots, y_n)$ of (7) with $|y(t_0)|$ small enough there exists a solution $x(t)$ of (6) with $x(t_0) = z(t_0)$ such that $x_i(t) = z_i(t) + (a_i + o(1))$. (B) If $r \leq 1$ then for every solution $y(t)$ of (7) there exists a solution $y(t)$ of (6) with $y(t_0) = z(t_0)$ such that $y_i(t) = z_i(t) + (a_i + o(1))\rho_i(t)$, $i = 1, 2, \ldots, n$, and conversely for every solution $x(t)$ of (6) there exists a solution $y(t)$ of (7) with $y(t_0) = x(t_0)$ such that $x_i(t) = y_i(t) + (a_i + o(1))\rho_i(t)$, $i = 1, 2, \ldots, n$.*

In particular, if there exists a solution of (7) such that $x_i(t) = z_i(t) + (a_i + o(1))\rho_i(t)$ with $(a_1, a_2, \ldots, a_n) \neq 0$ then there exists at least a solution $x(t)$ of (6) such that $\lim_{t\to\infty}(x_i(t)/\rho_i(t)) = b_i$, $i = 1, 2, \ldots, n$, with $(b_1, b_2, \ldots, b_n) \neq 0$.

Proof. A general solution of (6) can be written in the form

$$x(t) = y(t) + \int_{t_0}^{t} U(t)U^{-1}(s) f(s, x(s), x(u(s)))\,ds + \int_0^t U(t)U^{-1}(s) g(s)\,ds.$$

By [3, Lemma 3, p. 8] $U(t) \cdot U^{-1}(s) = C_{ir}(t, s)$ where

$$c_{ir}(t, s) = \frac{\det C'_{ir}(t, s)}{\det U(s)}$$

and $(C'_{ir}(t, s))$ is the matrix which we obtain by substitution of the row of order i of matrix $U(t)$ in the row of order r in the matrix $U(s)$. Then if $U(t) = (y_{il}(t))$, i, $l = 1, 2, \ldots, n$,

$$\begin{aligned} C'_{ir}(t, s) &= \sum_{l=1}^{n} y_{il}(t) \cdot G_{rl}(s) \\ &= \sum_{l=1}^{n} y_{il}(t)\rho_i(t)\rho_i(t)^{-1}\rho_r(s)^{-1} \det \rho(s) U_{rl}(s) \\ &= \rho_i(t)\rho_r(s)^{-1} \det \rho(s) \sum_{l=1}^{n} D_{rl}(t, s) \end{aligned}$$

where $D_{rl}(t, s)$ is a bounded function and $\lim_{t\to\infty} D_{rl}(t, s)$ exists, since $\lim_{t\to\infty} y_{il}(t)[\rho_i(t)]^{-1}$ exists by hypothesis.

By the Jacobi-Liouville formula we have

$$\det U(t) = k \exp\left(-\int_{T_0}^{t} T_r(A(s))\,ds\right), \qquad T_0 \geq t_0,$$

and

$$C_{ir}(t, s) = \frac{1}{k}\left[\exp -\int_{T_0}^{s} T_r(A(v))\,dv\right] \rho_i(t)\rho_r^{-1}(s) \det \rho(s) \sum_{l=1}^{n} D_{rl}(t, s),$$

then

$$\begin{aligned}\frac{x_i(t)}{\rho_i(t)} = \frac{y_i(t)}{\rho_i(t)} &+ \frac{1}{K}\int_{t_0}^{t} \det \rho(s)\left[\exp -\int_{T_0}^{s} T_r(A(v))\,dv\right] \sum_{r=1}^{n} \rho_r^{-1}(s) \sum_{l=1}^{n} D_{rl}(t, s) \\ &+ f_r(s, x(s), x(u(s)))\,ds \\ &+ \frac{1}{K}\int_{t_0}^{t} \det \rho(s)\left[\exp -\int_{T_0}^{s} T_r(A(v))\,dv\right] \sum_{j=1}^{n} \rho_j^{-1}(s) \sum_{l=1}^{n} D_{jl}(t, s) g_j(s)\end{aligned}$$

by substitution of $f_r(s, x(s), x(u(s)))$ given by hypothesis (H_6'); the reasoning is the same used in Theorem 1 and we conclude that $x_i(t) = y_i(t) + (a_i + o(1))\rho_i(t)$.

We have only to prove now that there exists a solution $x(t)$ for which $\lim_{t\to\infty}(x_i(t)/\rho_i(t)) = b_i$ with $b_i \neq 0$ for at least one i.

By hypothesis there exists at least a solution $y(t)$ of (7) for which $\lim_{t\to\infty}(y_i(t)/\rho_i(t)) = a_i$ with $a_i \neq 0$ for some i.

From the proof of Theorem 2 and [2, Lemma 1], it is not difficult to see that

$$\lim_{t\to\infty} \frac{1}{\rho_i(t)}\int_{t_0}^{t} \sum_{r=1}^{n} C_{ir}(t, s) f_r(s, x(s), x(u(s)))\,ds + \frac{1}{\rho_i(t)}\int_{t_0}^{t} \sum_{j=1}^{n} C_{ij}(t, s) h_j(s)\,ds = A_i$$

with $|A_i| < a_i$ if t_0 is large enough, then

$$\lim_{t\to\infty} (x_i(t)/\rho_i(t)) = a_i + A_i \neq 0.$$

To prove that, if $r \leq 1$, for every solution $x(t)$ of (6) there exists a solution $y(t)$ of (7) such that $x_i(t) = y_i(t) + (a_i + o(1))\rho_i(t)$, since every solution of (6) has this form, we have only to write $y(t)$ given by the integral equation

$$y(t) = x(t) - \int_{t_0}^{t} U(t)U^{-1}(s) f(s, x(s), x(u(s)))\,ds - \int_{t_0}^{t} U(t)U^{-1}(s) g(s)\,ds$$

As a simple application of Theorem 2 consider the equation

$$x^{(n)} + f(t, x(t), x^{(1)}(t) \dots x^{(n-1)}(t), x(u_1(t)), x^{(1)}(u_2(t)) \dots x^{(n-1)}(u_n(t))) = 0. \tag{8}$$

As every solution of $y^{(n)} = 0$ has the form

$$y^{(i)}(t) = C_{0i} + C_{1i}t + \dots + C_{n-i,i}t^{n-i},$$

$i = 0, 1, \dots, n-1$, every solution of (8) has the form

$$x^{(i)}(t) = C_{0i} + C_{1i} + \dots + C_{n-i,i}t^{n-i} + (a_i + o(1))t^{n-i},$$

$$\rho_i(t) = t^{n-i}, \qquad i = 0, 1, \dots, n-1.$$

In particular there is a nonoscillatory solution of (8) satisfying $\lim_{t\to\infty}(x^{(n-i)}(t)/t^{n-i}) = C_{n-i,i} \neq 0$ for every i, and this result generalizes Theorem 1 of [5] and Theorem 1 of [6]. Particularly if (7) is the general singular system considered in [3] then equation (6) which in this case contains (8) will have a solution satisfying $\lim_{t\to\infty}(x_i(t)/t^{a i}) = b_i$.

Example. The differential equation

$$s'(t) = -\beta(t)s(t)[2\gamma(t) + s(t-14) - s(t-12)] + \gamma(t) \tag{9}$$

is a delay differential equation, studied by J. A. Yorke [10] which describes how the measles spreads through a population as a function of time, where $s(t)$ is the number of susceptible individuals, $\beta(t)$ is a proportionality constant that is seasonally dependent, having a one-year period, and $\gamma(t)$ is the rate at which susceptible individuals enter the population (through net immigration or birth). If $I(t)$ is the number of infectious individuals,

$$I(t) = \int_{t-14}^{t-12} \beta(x)s(x)I(x)\,dx = \int_{t-14}^{t-12} [\gamma(x) - s'(x)]\,dx.$$

Equation (9) can be written in the form

$$s'(t) = -2\beta(t)\gamma(t)s(t) - \beta(t)[s(t-14) - s(t-12)]s(t) + \gamma(t). \tag{10}$$

Then if $\rho(t) = \exp -2\int_0^t \beta(t)\gamma(t)\,dt$, $u_1(t) = t - 14$, $u_2(t) = t - 12$,

$$|f(t, x, y)| = |-\beta(t)[s(t-14) - s(t-12)]s(t)|$$

$$\leq |\beta(t)[|s(t)|^2 + \tfrac{1}{2}|s(t-14)|^2 + \tfrac{1}{2}|s(t-12)|^2],$$

$g(t) = \gamma(t)$ under certain circumstances, it is possible to have hypotheses (H_4') and (H_5') satisfied, that is:

$$\int^\infty \beta(t)\exp\left[-2\int_0^t \beta(\tau)\gamma(\tau)\,d\tau\right]dt < \infty,$$

$$\int^\infty \beta(t)\exp\left[-2\int_0^{t-14} \beta(\tau)\gamma(\tau)\,d\tau + 2\int_0^t \beta(\tau)\gamma(\tau)\,d\tau\right]dt < \infty,$$

$$\int^\infty \beta(t)\exp\left[-2\int_0^{t-12} \beta(\tau)\gamma(\tau)\,d\tau + 2\int_0^t \beta(\tau)\gamma(\tau)\,d\tau\right]dt < \infty,$$

$$\int^{\infty} \gamma(t) \exp\left[2 \int_0^t \beta(\tau)\gamma(\tau)\, d\tau\right] dt < \infty.$$

Under these conditions by Theorem 2 there is a solution for $s(0)$ small enough such that

$$s(t) = \exp\left[-2 \int_0^t \gamma(\tau)\beta(\tau)\, d\tau\right] [a + o(1)].$$

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INSTITUTO DE CIÊNCIAS MATEMÁTICAS, DE SAO CARLOS, UNIVERSIDADE DE SÃO PAULO, SÃO CARLOS, BRASIL

DEPARTMENT OF MATHEMATICS, BROWN UNIVERSITY, PROVIDENCE, RHODE ISLAND 02912

FACULDADE DE FILOSOFIA, CIÊNCIAS E LETRAS DE RIBEIRÃO PRETO, RIBEIRÃO PRETO, BRASIL