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INTERPOLATION BY TRANSFORMS OF DISCRETE MEASURES

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ABSTRACT. Let G be a compact abelian group, and Γ its character group. Given $E \subset \Gamma$, E^a denotes the set of all accumulation points of E in $\overline{\Gamma}$, the Bohr compactification of Γ. In this paper it is shown that the inclusion $(L^1(G))\hat{}|_E \subset (l^1(G))\hat{}|_E$ obtains if and only if $E \cap E^a = \emptyset$ and there exists a measure $\mu \in M(G)$ such that $\hat{\mu} = 1$ on E and $\hat{\mu} = 0$ on $\Gamma \cap E^a$.

Let G be an infinite compact abelian group, Γ its character group, and $A(\Gamma) = (L^1(G))\hat{}$ the Fourier algebra on Γ. For any subset E of Γ, we denote by $A(E)$ the restriction algebra $A(\Gamma)|_E = A(\Gamma)/I(E)$ with the quotient norm, where $I(E) = \{f \in A(\Gamma): f = 0 \text{ on } E\}$. Similarly, we consider $A_d(\Gamma) = (l^1(G))\hat{}$ and $A_d(E) = A_d(\Gamma)|_E$. Thus $A_d(E)$ is isometrically isomorphic to $A(\overline{E}) = A(\overline{\Gamma})|_{\overline{E}}$, where $\overline{E}$ denotes the closure of E in $\overline{\Gamma}$, the Bohr compactification of Γ. Notice that $A(\Gamma) \subset C_0(\Gamma)$ but $A_d(\Gamma) \cap C_0(\Gamma) = \{0\}$ (see [4, 5.6.9]).

Interpolation by transforms of discrete measures has been studied by many authors; the reader is referred in particular to [1], [2], and [3]. In this paper we prove the following

Theorem. *Let $E \subset \Gamma$, and E^a the set of all accumulation points of E in $\overline{\Gamma}$. The inclusion $A(E) \subset A_d(E)$ obtains if and only if $E \cap E^a = \emptyset$ and there exists a measure $\mu \in M(G)$ such that $\hat{\mu} = 1$ on E and $\hat{\mu} = 0$ on $\Gamma \cap E^a$.*

To prove this, we need a

Lemma. *Let $E \subset \Gamma$, and $K = \Gamma \cap E^a$. Then $\{f \in A(E \cup K): f = 0 \text{ on } K\} \subset A_d(E \cup K)$ isometrically.*

Proof. Put $F = E \cup K = \overline{E} \cap \Gamma$ and $E_0 = E \backslash K$. Notice that each point of E_0 is isolated in $\overline{E}$.

Take any $f \in A(F)$ with $f = 0$ on K. We must prove that

$$f \in A_d(F) \quad \text{and} \quad \|f\|_{A_d(F)} = \|f\|_{A(F)}. \tag{1}$$

It suffices to confirm this assuming that f has finite support.

Choose any pseudomeasure $\psi \in A'(\overline{E}) = I(\overline{E})^\perp$ with $\|\psi\|_{PM} \leq 1$. Using

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Theorem 1 in [5] and its proof, we can prove the following: Given any finite subset E_1 of E_0 and any $\epsilon > 0$, there exists a finite subset E_2 of Γ, with $E_1 \subset E_2$, such that to each neighborhood V of $0 \in \overline{\Gamma}$ there corresponds $h_V \in A(\overline{\Gamma})$ such that

$$\|h_V\|_{A(\overline{\Gamma})} < 1 + \epsilon, \quad \operatorname{supp}(h_V) \subset E_2 + V, \quad \text{and} \quad h_V = 1 \quad \text{on } E_1.$$

Then we have $\|h_V\psi\|_{PM} < 1 + \epsilon$, $\operatorname{supp}(h_V\psi) \subset \overline{E} \cap (E_2 + \overline{V})$, and $(h_V\psi)(\gamma) = \psi(\gamma)$ for all $\gamma \in E_1$ (notice that each point of E_0 is isolated in $\overline{E}$). Letting V converge to $0 \in \overline{\Gamma}$, we obtain a measure $P \in l^1(\Gamma)$ such that

$$(2) \quad \|P\|_{PM} \leq 1 + \epsilon, \quad \operatorname{supp}(P) \subset \overline{E} \cap E_2 \subset F, \quad \text{and} \quad P = \psi \quad \text{on } E_1.$$

Since $E_1 \subset E_0$ and $\epsilon > 0$ are arbitrary, we may pass to a weak-* limit in $A'(F) = I(F)^{\perp}$ and find a $\phi \in A'(F)$ such that $\|\phi\|_{PM} \leq 1$ and $\phi(\gamma) = \psi(\gamma)$ for all $\gamma \in E_0$. Since $f \in A(F)$ is supported by a finite subset of E_0, we have $f \in A_d(F) = A(\overline{E})$ and

$$\langle f, \psi\rangle = \sum_{\gamma \in E_0} f(\gamma)\psi(\gamma) = \sum_{\gamma \in E_0} f(\gamma)\phi(\gamma) = \langle f, \phi\rangle,$$

so that $|\langle f, \psi\rangle| \leq \|f\|_{A(F)}$. Since $\psi \in A'(\overline{E})$ is an arbitrary element with norm ≤ 1, the last inequality, combined with the Hahn-Banach theorem implies $\|f\|_{A_d(F)} \leq \|f\|_{A(F)}$. We also have

$$\|f\|_{A(F)} = \sup\{|\langle f, P\rangle|: P \in l^1(F), \|P\|_{PM} \leq 1\} \leq \|f\|_{A_d(F)},$$

which establishes (1). This completes the proof.

Proof of Theorem. Suppose $A(E) \subset A_d(E)$. Since $A(E) \subset \mathbf{C}_0(E)$, we then have

$$(3) \qquad A(E) \subset A_d(E) \cap \mathbf{C}_0(E) = \{f|_E: f \in A(\overline{E}), f = 0 \text{ on } E^a\}.$$

Therefore it is obvious that $E \cap E^a = \emptyset$. If $F = \overline{E} \cap \Gamma$ and $\xi_E \in l^\infty(F)$ is the characteristic function of E, then $\xi_E A(F) \subset B(F) = (M(G))\hat{}|_F$. In fact, $f \in A(F)$ implies $f|_E \in A(E)$. It follows from (3) that there exists $g \in A_d(\Gamma)$ such that $g = 0$ on $E^a \cap \Gamma$ and $g = f$ on E. Hence $\xi_E f = g|_F \in A_d(F)$. Since Γ is discrete, we conclude that $\xi_E \in B(F)$.

Conversely, suppose $E \cap E^a = \emptyset$ and $\xi_E \in B(F)$. Given $f \in A(E)$, let $g \in A(F)$ be any extension of f. Then $\xi_E g \in A(F)$, so that $\xi_E g \in A_d(F)$ by the Lemma. Hence $f = (\xi_E g)|_E \in A_d(E)$.

Remarks. A nontrivial example of a set $E \subset \Gamma$ with the property that E^a is a set of synthesis in $\overline{\Gamma}$ can be found in [6]. If $A(E) \subset A_d(E)$ and E^a is a set of synthesis for the algebra $A(\overline{\Gamma}) = A_d(\Gamma)$, then $A(E) = A_d(E) \cap \mathbf{C}_0(E)$. If E^a is a set of synthesis in $\overline{\Gamma}$, then

$$\{f \in A(E \cup K): f = 0 \text{ on } K\} = A_d(E \cup K) \cap \mathbf{C}_0(E \cup K).$$

This follows from the fact that $A_d(F) \cap C_0(F)$ is isometrically isomorphic to $\{f \in A(\bar{E}): f = 0 \text{ on } E^a\}$.

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