INTERPOLATION BY TRANSFORMS OF DISCRETE MEASURES

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ABSTRACT. Let $G$ be a compact abelian group, and $\Gamma$ its character group. Given $E \subset \Gamma$, $E^a$ denotes the set of all accumulation points of $E$ in $\overline{\Gamma}$, the Bohr compactification of $\Gamma$. In this paper it is shown that the inclusion $(L^1(G))^\wedge|_E \subset (L^1(G))^\wedge|_\Gamma$ obtains if and only if $E \cap E^a = \emptyset$ and there exists a measure $\mu \in M(G)$ such that $\hat{\mu} = 1$ on $E$ and $\hat{\mu} = 0$ on $\Gamma \cap E^a$.

Let $G$ be an infinite compact abelian group, $\Gamma$ its character group, and $A(\Gamma) = (L^1(G))^\wedge$ the Fourier algebra on $\Gamma$. For any subset $E$ of $\Gamma$, we denote by $A(E)$ the restriction algebra $A(\Gamma)|_E = A(\Gamma)/I(E)$ with the quotient norm, where $I(E) = \{ f \in A(\Gamma) : f = 0 \text{ on } E \}$. Similarly, we consider $A_d(\Gamma) = (L^1(G))^\wedge$ and $A_d(E) = A_d(\Gamma)|_E$. Thus $A_d(E)$ is isometrically isomorphic to $A(E) = A(\Gamma)|_\overline{E}$, where $\overline{E}$ denotes the closure of $E$ in $\overline{\Gamma}$, the Bohr compactification of $\Gamma$. Notice that $A(\Gamma) \subset C_0(\Gamma)$ but $A_d(\Gamma) \cap C_0(\Gamma) = \{ 0 \}$ (see [4, 5.6.9]).

Interpolation by transforms of discrete measures has been studied by many authors; the reader is referred in particular to [1], [2], and [3]. In this paper we prove the following

Theorem. Let $E \subset \Gamma$, and $E^a$ the set of all accumulation points of $E$ in $\overline{\Gamma}$. The inclusion $A(E) \subset A_d(E)$ obtains if and only if $E \cap E^a = \emptyset$ and there exists a measure $\mu \in M(G)$ such that $\hat{\mu} = 1$ on $E$ and $\hat{\mu} = 0$ on $\Gamma \cap E^a$.

To prove this, we need a

Lemma. Let $E \subset \Gamma$, and $K = \Gamma \cap E^a$. Then $\{ f \in A(E \cup K) : f = 0 \text{ on } K \} \subset A_d(E \cup K)$ isometrically.

Proof. Put $F = E \cup K = \overline{E} \cap \Gamma$ and $E_0 = E \setminus K$. Notice that each point of $\overline{E_0}$ is isolated in $\overline{E}$.

Take any $f \in A(F)$ with $f = 0$ on $K$. We must prove that

$$f \in A_d(F) \quad \text{and} \quad \|f\|_{A_d(F)} = \|f\|_{A(F)}$$

It suffices to confirm this assuming that $f$ has finite support.

Choose any pseudomeasure $\psi \in A'(\overline{E}) = I(\overline{E})^\perp$ with $\|\psi\|_{PM} \leq 1$. Using
Theorem 1 in [5] and its proof, we can prove the following: Given any finite subset $E_1$ of $E_0$ and any $\varepsilon > 0$, there exists a finite subset $E_2$ of $\Gamma$, with $E_1 \subseteq E_2$, such that to each neighborhood $V$ of $0 \in \Gamma$ there corresponds $h_v \in A(\Gamma)$ such that

$$\|h_v\|_{A(\Gamma)} < 1 + \varepsilon, \quad \text{supp}(h_v) \subseteq E_2 + V, \quad \text{and} \quad h_v = 1 \quad \text{on} \quad E_1.$$

Then we have $\|h_v\psi\|_{P^\infty} < 1 + \varepsilon$, $\text{supp}(h_v\psi) \subseteq \overline{E} \cap (E_2 + \overline{V})$, and $(h_v\psi)(y) = \psi(y)$ for all $y \in E_1$ (notice that each point of $E_0$ is isolated in $\overline{E}$). Letting $V$ converge to $0 \in \Gamma$, we obtain a measure $P \in l^1(\Gamma)$ such that

$$(2) \|P\|_{P^\infty} \leq 1 + \varepsilon, \quad \text{supp}(P) \subseteq \overline{E} \cap E_2 \subseteq F, \quad \text{and} \quad P = \psi \quad \text{on} \quad E_1.$$

Since $E_1 \subseteq E_0$ and $\varepsilon > 0$ are arbitrary, we may pass to a weak-* limit in $A'(F) = l(F)^\perp$ and find a $\phi \in A'(F)$ such that $\|\phi\|_{P^\infty} \leq 1$ and $\phi(y) = \psi(y)$ for all $y \in E_0$. Since $f \in A(F)$ is supported by a finite subset of $E_0$, we have $f \in A_d(F) = A(\overline{E})$ and

$$\langle f, \psi \rangle = \sum_{y \in E_0} f(y)\psi(y) = \sum_{y \in E_0} f(y)\phi(y) = \langle f, \phi \rangle,$$

so that $|\langle f, \psi \rangle| \leq \|f\|_{A(F)}$. Since $\psi \in A'(\overline{E})$ is an arbitrary element with norm $\leq 1$, the last inequality, combined with the Hahn-Banach theorem implies $\|f\|_{A_d(F)} \leq \|f\|_{A(F)}$. We also have

$$\|f\|_{A_d(F)} = \sup\{\|f, P\|: P \in l^1(F), \|P\|_{P^\infty} \leq 1\} \leq \|f\|_{A_d(F)},$$

which establishes (1). This completes the proof.

Proof of Theorem. Suppose $A(E) \subseteq A_d(E)$. Since $A(E) \subseteq C_0(E)$, we then have

$$(3) \quad A(E) \subseteq A_d(E) \cap C_0(E) = \{f|_E: f \in A(\overline{E}), f = 0 \quad \text{on} \quad E^a\}.$$

Therefore it is obvious that $E \cap E^a = \emptyset$. If $F = \overline{E} \cap \Gamma$ and $\xi_E \in l^\infty(F)$ is the characteristic function of $E$, then $\xi_E A(F) \subseteq B(F) = (\theta(G)\Gamma)|_F$. In fact, $f \in A(F)$ implies $f|_E \in A(E)$. It follows from (3) that there exists $g \in A_d(\Gamma)$ such that $g = 0$ on $E^a \cap \Gamma$ and $g = f$ on $E$. Hence $\xi_E f = g|_F \in A_d(F)$. Since $\Gamma$ is discrete, we conclude that $\xi_E \in B(F)$.

Conversely, suppose $E \cap E^a = \emptyset$ and $\xi_E \in B(F)$. Given $f \in A(E)$, let $g \in A(F)$ be any extension of $f$. Then $\xi_E\xi_E g \in A(F)$, so that $\xi_E\xi_E g \in A_d(F)$ by the Lemma. Hence $f = (\xi_E\xi_E g)|_F \in A_d(E)$.

Remarks. A nontrivial example of a set $E \subseteq \Gamma$ with the property that $E^a$ is a set of synthesis in $\overline{\Gamma}$ can be found in [6]. If $A(E) \subseteq A_d(E)$ and $E^a$ is a set of synthesis for the algebra $\overline{A(\Gamma)} = A_d(\Gamma)$, then $A(E) = A_d(E) \cap C_0(E)$.

If $E^a$ is a set of synthesis in $\overline{\Gamma}$, then

$$\{f \in A(E \cup K): f = 0 \quad \text{on} \quad K\} = A_d(E \cup K) \cap C_0(E \cup K).$$
This follows from the fact that $A_d(F) \cap C_0(F)$ is isometrically isomorphic to \{$f \in A(E) : f = 0$ on $E^a$\}.

REFERENCES


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