ON A THEOREM OF KÔMURA-KOSHI AND OF ANDÔ-ELLIS

YAU-CHUEN WONG

ABSTRACT. Komura and Koshi's result, which states that the topology \( \mathcal{T} \) of a nuclear locally convex vector lattice \((E, C, \mathcal{T})\) is the topology \( o(E, E') \) of uniform convergence on all order-intervals in \( E' \), is generalized to the case when \((E, C, \mathcal{T})\) is only a locally solid space. Andô-Ellis' theorem, concerning the duality of strict \( B \)-cones and normality in normed vector spaces, is generalized to the metrizable case.

1. Introduction. By a locally solid space we mean an ordered convex space \((E, C, \mathcal{T})\) such that \( \mathcal{T} \) admits a neighbourhood base at 0 consisting of convex and solid sets in \( E \). Locally solid spaces share a number of important properties with locally convex vector lattices (for example, see Walsh [9] and [10]).

Komura and Koshi [4] have shown that the topology \( \mathcal{T} \) of a nuclear locally convex vector lattice \((E, C, \mathcal{T})\) is the topology \( o(E, E') \) of uniform convergence on all order-intervals in \( E' \). This result is generalized to the case when \((E, C, \mathcal{T})\) is a locally solid space.

Andô-Ellis' theorem, which states that for a normed space \((E, \|\cdot\|)\) ordered by a complete cone \( C \) the normality of \( C' \) in \((E', \|\cdot\|)\) implies that \( C \) be a strict \( B \)-cone in \((E, \|\cdot\|)\), is one of the important results in the theory of ordered normed spaces. We shall give, in the final section, a generalization on Andô-Ellis' theorem to the metrizable case.

2. A generalization of Komura and Koshi's theorem. Let \((E, C, \mathcal{T})\) be a locally solid space (\( C \) is binormal for \( \mathcal{T} \) in the sense of Walsh [9]) whose dual is denoted by \( E' \), and suppose that \( C' \) is the dual cone of \( C \). Then the weak topology \( o(E, E') \) is locally \( o \)-convex (i.e., \( C \) is normal in \((E, o(E, E'))\)), and it is generated by the family \( \{p_f : f \in C'\} \) of monotone seminorms, where each \( p_f \) is defined by \( p_f(x) = |f(x)| \) (\( x \in E \)). Suppose \( V_f = \{x \in E : p_f(x) \leq 1\} \) and \( S(V_f) = \bigcup [-u, u] : u \in V_f \cap C \}. Walsh [9, (1.3.7)] has shown that the gauge function of \( S(V_f) \), denoted by \( p_{f,S} \), is given by

\[
(1) \quad p_{f,S}(x) = \inf \{p_f(y) : y \in C, -y \leq x \leq y\} = \inf \{|f(x)| : y \in C, -y \leq x \leq y\}
\]

therefore the family \( \{p_{f,S} : f \in C'\} \) of seminorms generates the locally solid...
topology $\sigma_s(E, E')$ associated with $\sigma(E, E')$. As $p_f(u) = p_{f, s}(u)$ for all $u \in C$, it follows that $p_{f, s}$ is additive on $C$.

For any $f \in C'$, as the topological dual $E'$ of $(E, C, \mathcal{F})$ is a solid subspace of $E^*$ (see [10, (6.5)]), it follows that $[-f, f] \subset E'$; we now define

$$(2) \quad q_f(x) = \sup \{g(x): -f \leq g \leq f\} \quad (x \in E).$$

It is known from Peressini [6, p. 130] that $\{q_f: f \in C'\}$ generates the topology $\sigma(E, E')$ of uniform convergence on all order-intervals in $E'$.

It is not hard to see that $q_f(x) \leq p_{f, s}(x) \quad (x \in E)$. On the other hand, if $g$ is in the polar of $\{x \in E: p_{f, s}(x) \leq 1, \, \text{then} \, |g(u)| \leq p_{f, s}(u) = f(u) \quad (u \in C)$, so $-f \leq g \leq f$, and thus $p_{f, s}(x) \leq q_f(x) \quad (x \in E)$. This remark makes the following result clear.

**Lemma 1.** Let $(E, C, \mathcal{F})$ be a locally solid space. For any $f \in C'$, we have $p_{f, s} = q_f$; consequently, $\sigma_s(E, E') = \sigma(E, E')$.

**Theorem 2.** For a nuclear locally solid space $(E, C, \mathcal{F})$, $\mathcal{F}$ coincides with $\sigma(E, E')$.

**Proof.** This theorem can be deduced from Walsh’s result [9, (3.2.5)] and the polar characterization of local decomposability (see [9, (1.3.3)] or [10, (3.10)]); but we present here a somewhat more direct and elementary proof.

Let $U$ be any closed, convex, circled $\mathcal{F}$-neighbourhood of 0 and suppose that $p_U$ is the gauge of $U$. The nuclearity of $E$ insures that there exists a convex solid $\mathcal{F}$-neighbourhood $V$ of 0, a sequence $(g_n)$ in the polar $V^0$ of $V$, and $(\lambda_n) \in l_1$ such that

$$(3) \quad p_U(x) \leq \sum_n |\lambda_n g_n(x)| \quad (x \in E).$$

Since $E'$ is a solid subspace of $E^*$, by a theorem of Jameson (see [3, (1.7.1)] or [10, (1.17)]), $V^0$ is a solid subset of $E'$, hence for any $g_n$, there exists $f_n \in V^0 \cap C'$ such that $-f_n \leq g_n \leq f_n$. Define $f(x) = \sum_n |\lambda_n| f_n(x) \quad (x \in E)$. Clearly $f$ is a positive continuous linear functional on $E$. On the other hand, the inequality (3) insures that

$\inf\{f(y): \gamma \in C, \gamma \leq -x \leq \gamma\} = p_{f, s}(x) \quad (x \in E),$

hence $\mathcal{F}$ is coarser than $\sigma_s(E, E')$. As $\sigma_s(E, E')$ is the smallest locally solid topology finer than $\sigma(E, E')$, we conclude that $\mathcal{F} = \sigma_s(E, E')$, and hence from Lemma 1 that $\mathcal{F} = \sigma(E, E')$.

The preceding result was proved by Komura and Koshi [4] in the special case when $(E, C, \mathcal{F})$ is a locally convex vector lattice.

A sequence $(x_n)$ in an ordered convex space $(E, C, \mathcal{F})$ is said to be positive if $x_n \in C$ for all $n$. Let $(E, C, \mathcal{F})$ be a locally solid space and $N$ the set of all natural numbers. We denote by $l^1[N, E]$ (resp. $l^1(N, E)$) the
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space consisting of all absolutely summable (resp. summable) sequences in $E$. It is not hard to see that if $(E, C, \bar{J})$ is metrizable, then $l^1[N, E]$ is a solid subspace of the (ordered) product space $(E^N, C^N)$.

**Corollary 3.** A metrizable locally solid space $(E, C, \bar{J})$ is nuclear if and only if it satisfies the following conditions:

(a) $\bar{J} = o(E, E')$;

(b) each summable sequence in $E$ is the difference of two positive summable sequences in $E$.

**Proof.** Suppose that $E$ is nuclear. Then $\bar{J} = o(E, E')$ by Theorem 2, and $l^1(N, E) = l^1[N, E]$ in view of a well-known result. According to the remark preceding the corollary, $l^1[N, E]$ is a solid subspace of $(E^N, C^N)$, hence (b) holds; therefore the condition is necessary. To see that it is also sufficient, we first note that each positive summable sequence in $(E, C, o(E, E'))$ is absolutely summable with respect to the topology $o(E, E')$. Therefore $l^1(N, E) = l^1[N, E]$. As $E$ is metrizable, we conclude from Pietsch [8, (4.2.5)] that $E$ is nuclear.

For a base normed space $(E, C, \|\cdot\|)$, the norm $\|\cdot\|$ is additive on $C$, it follows from Walsh [9, (3.2.5)] that the norm topology coincides with $o(E, E')$. But a normed space with infinite dimension is not nuclear, therefore condition (b) in Corollary 3 is essential.

3. A generalization of Andô-Ellis' theorem. For a subset $V$ of $(E, C)$, we define

$$D(V) = \{x \in V: x = \lambda u - (1 - \lambda)w, \lambda \in [0, 1] \text{ and } u, w \in V \cap C\}.$$ 

It is clear that if $V$ is convex and circled, then $D(V) = \text{co}(V \cap C) \cup (V \cap C)$ and

$$V \cap C = V \cap C$$

A subset $V$ of $(E, C)$ is said to be decomposable if $V = D(V)$. An ordered convex space $(E, C, \bar{J})$ is called a locally decomposable space (C is conormal in $(E, \bar{J})$ in the sense of Walsh [9]) if $\bar{J}$ admits a neighbourhood base at 0 consisting of convex decomposable sets in $E$. According to formula (4), an ordered convex space $(E, C, \bar{J})$ is locally decomposable if and only if $C$ gives an open decomposition in $(E, \bar{J})$ in the sense of Jameson [3, p. 94]. The topological dual $E'$ of a locally decomposable space $(E, C, \bar{J})$ is an order-convex subspace of $E^*$. In this section $\beta(E', E)$ denotes the strong topology on $E'$ while $o(E', E)$ is the weak topology on $E'$.

**Theorem 4.** Let $(E, C, \bar{J})$ be an infrabarrelled ordered convex space with a countable fundamental system of $\bar{J}$-bounded subsets of $E$. If $(E, C, \bar{J})$
is locally decomposable then \((E', C', \beta(E', E))\) is locally \(o\)-convex.

**Proof.** According to the hypotheses, \((E', C', \beta(E', E))\) is metrizable, therefore it is sufficient to show, by [10, (5.2)], that the order-convex hull in \(E'\) of each \(\beta(E', E)\)-bounded subset \(B\) of \(E'\) is \(\beta(E', E)\)-bounded. Let \(B\) be such a set. \(B\) is \(\overline{\mathcal{F}}\)-equicontinuous, it follows from the polar characterization of local decomposability (see Walsh [9, (1.3.3)] or [10, (3.10)]) that \((B + C^*) \cap (B - C^*)\) is \(\overline{\mathcal{F}}\)-equicontinuous; consequently, the order-convex hull of \(B\) is \(\beta(E', E)\)-bounded.

**Corollary 5.** Let \((E, C, \overline{\mathcal{F}})\) be an infrabarrelled ordered convex space with a countable fundamental system of bounded sets. If \((E, C, \overline{\mathcal{F}})\) is locally solid then so is \((E', C', \beta(E', E))\).

**Proof.** It is known from [10, (6.3)] that an ordered convex space is locally solid if and only if it is both locally \(o\)-convex and locally decomposable. Now Theorem 4 insures that \((E', C', \beta(E', E))\) is locally \(o\)-convex. On the other hand, by Schaefer's theorem [8, p. 220], the local \(o\)-convexity of \(\overline{\mathcal{F}}\) implies that \(C'\) is a strict \(\overline{\mathcal{B}}\)-cone in \((E', \beta(E', E))\); by Jameson [3, (3.3.6)] (or [9, (1.2.4)]), \((E', C', \beta(E', E))\) is locally decomposable because \((E', \beta(E', E))\) is metrizable and surely bornological.

The following result should be compared with Jameson [3, (3.5.4)].

**Theorem 6.** Let \((E, C, \overline{\mathcal{F}})\) be a metrizable ordered convex space such that \(C\) is \(\overline{\mathcal{F}}\)-complete. If \((E', C', \beta(E', E))\) is locally \(o\)-convex then \((E, C, \overline{\mathcal{F}})\) is locally decomposable, and hence \(E\) is \(\overline{\mathcal{F}}\)-complete.

**Proof.** For any convex circled \(\overline{\mathcal{F}}\)-neighbourhood \(V\) of 0, it is known from [10, (2.11)] that \((D(V))^0 = (V^0 + C^*) \cap (V^0 - C^*) \cap E'\), therefore \((D(V))^0\) is the order-convex hull in \(E'\) of \(V^0\). As \(E\) is metrizable and surely infrabarrelled, it follows from the local \(o\)-convexity of \((E', C', \beta(E', E))\) that \(D(V)\) is a \(\overline{\mathcal{F}}\)-neighbourhood of 0. As \(C\) is complete, we conclude from the open mapping theorem (see Ng [5, Theorem 2]) that \((E, C, \overline{\mathcal{F}})\) is locally decomposable.

Finally, since \((E, C, \overline{\mathcal{F}})\) is locally decomposable for which \(C\) is \(\overline{\mathcal{F}}\)-complete, it follows from Klee's theorem (see [10, (3.6)]) that \(E\) must be complete. Therefore the proof is complete.

The preceding theorem, together with a result of Schaefer [8, p. 220], shows that for a metrizable ordered convex space \((E, C, \overline{\mathcal{F}})\) such that \(C\) is \(\overline{\mathcal{F}}\)-complete, if \(\beta(E', E)\) is locally \(o\)-convex then so is \(\sigma(E', E)\).

Clearly an ordered normed space \((E, C, \|\cdot\|)\) is locally decomposable if and only if \(C\) is a strict \(\overline{\mathcal{B}}\)-cone in \((E, \|\cdot\|)\). Therefore Theorems 4 and 6 constitute a generalization of Andô-Ellis' theorem [1], [2] and [5, Theorem 3].

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REFERENCES


DEPARTMENT OF MATHEMATICS, YALE UNIVERSITY, NEW HAVEN, CONNECTICUT
06520

UNITED COLLEGE, THE CHINESE UNIVERSITY OF HONG KONG, HONG KONG