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ON A THEOREM OF KŌMURA-KOSHI AND OF ANDÔ-ELLIS

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ABSTRACT. Kōmura and Koshi's result, which states that the topology $\mathfrak{T}$ of a nuclear locally convex vector lattice $(E, C, \mathfrak{T})$ is the topology $o(E, E')$ of uniform convergence on all order-intervals in E', is generalized to the case when $(E, C, \mathfrak{T})$ is only a locally solid space. Andô-Ellis' theorem, concerning the duality of strict $\mathfrak{B}$-cones and normality in normed vector spaces, is generalized to the metrizable case.

1. **Introduction.** By a locally solid space we mean an ordered convex space $(E, C, \mathfrak{T})$ such that $\mathfrak{T}$ admits a neighbourhood base at 0 consisting of convex and solid sets in E. Locally solid spaces share a number of important properties with locally convex vector lattices (for example, see Walsh [9] and [10]).

Kōmura and Koshi [4] have shown that the topology $\mathfrak{T}$ of a nuclear locally convex vector lattice $(E, C, \mathfrak{T})$ is the topology $o(E, E')$ of uniform convergence on all order-intervals in E'. This result is generalized to the case when $(E, C, \mathfrak{T})$ is a locally solid space.

Andô-Ellis' theorem, which states that for a normed space $(E, \|\cdot\|)$ ordered by a complete cone C the normality of C' in $(E', \|\cdot\|)$ implies that C be a strict $\mathfrak{B}$-cone in $(E, \|\cdot\|)$, is one of the important results in the theory of ordered normed spaces. We shall give, in the final section, a generalization on Andô-Ellis' theorem to the metrizable case.

2. **A generalization of Kōmura and Koshi's theorem.** Let $(E, C, \mathfrak{T})$ be a a locally solid space (C is binormal for $\mathfrak{T}$ in the sense of Walsh [9]) whose dual is denoted by E', and suppose that C' is the dual cone of C. Then the weak topology $\sigma(E, E')$ is locally o-convex (i.e., C is normal in $(E, \sigma(E, E')))$, and it is generated by the family $\{p_f: f \in C'\}$ of monotone seminorms, where each p_f is defined by $p_f(x) = |f(x)|$ $(x \in E)$. Suppose $V_f = \{x \in E: p_f(x) \leq 1\}$ and $S(V_f) = \bigcup\{[-u, u]: u \in V_f \cap C\}$. Walsh [9, (1.3.7)] has shown that the gauge function of $S(V_f)$, denoted by $p_{f,S}$, is given by

$$(1) \quad p_{f,S}(x) = \inf\{p_f(y): y \in C, -y \leq x \leq y\} = \inf\{f(y): y \in C, -y \leq x \leq y\}$$

therefore the family $\{p_{f,S}: f \in C'\}$ of seminorms generates the locally solid

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topology $\sigma_S(E, E')$ associated with $\sigma(E, E')$. As $p_f(u) = p_{f,S}(u)$ for all $u \in C$, it follows that $p_{f,S}$ is additive on C.

For any $f \in C'$, as the topological dual E' of $(E, C, \mathfrak{T})$ is a solid subspace of E^* (see [10, (6.5)]), it follows that $[-f, f] \subset E'$; we now define

$$q_f(x) = \sup\{g(x): -f \leq g \leq f\} \quad (x \in E). \tag{2}$$

It is known from Peressini [6, p. 130] that $\{q_f: f \in C'\}$ generates the topology $o(E, E')$ of uniform convergence on all order-intervals in E'.

It is not hard to see that $q_f(x) \leq p_{f,S}(x)$ $(x \in E)$. On the other hand, if g is in the polar of $\{x \in E: p_{f,S}(x) \leq 1\}$, then $|g(u)| \leq p_{f,S}(u) = f(u)$ $(u \in C)$, so $-f \leq g \leq f$, and thus $p_{f,S}(x) \leq q_f(x)$ $(x \in E)$. This remark makes the following result clear.

Lemma 1. *Let $(E, C, \mathfrak{T})$ be a locally solid space. For any $f \in C'$, we have $p_{f,S} = q_f$; consequently, $\sigma_S(E, E') = o(E, E')$.*

Theorem 2. *For a nuclear locally solid space $(E, C, \mathfrak{T})$, $\mathfrak{T}$ coincides with $o(E, E')$.*

Proof. This theorem can be deduced from Walsh's result [9, (3.2.5)] and the polar characterization of local decomposability (see [9, (1.3.3)] or [10, (3.10)]); but we present here a somewhat more direct and elementary proof. Let U be any closed, convex, circled $\mathfrak{T}$-neighbourhood of 0 and suppose that p_U is the gauge of U. The nuclearity of E insures that there exists a convex solid $\mathfrak{T}$-neighbourhood V of 0, a sequence (g_n) in the polar V^0 of V, and $(\lambda_n) \in l_1$ such that

$$p_U(x) \leq \sum_n |\lambda_n g_n(x)| \quad (x \in E). \tag{3}$$

Since E' is a solid subspace of E^*, by a theorem of Jameson (see [3, (1.7.1)] or [10, (1.17)]), V^0 is a solid subset of E', hence for any g_n, there exists $f_n \in V^0 \cap C'$ such that $-f_n \leq g_n \leq f_n$. Define $f(x) = \sum_n |\lambda_n| f_n(x)$ $(x \in E)$. Clearly f is a positive continuous linear functional on E. On the other hand, the inequality (3) insures that

$$P_U(x) \leq \inf\{f(y): y \in C, -y \leq x \leq y\} = p_{f,S}(x) \quad (x \in E),$$

hence $\mathfrak{T}$ is coarser than $\sigma_S(E, E')$. As $\sigma_S(E, E')$ is the smallest locally solid topology finer than $\sigma(E, E')$, we conclude that $\mathfrak{T} = \sigma_S(E, E')$, and hence from Lemma 1 that $\mathfrak{T} = o(E, E')$.

The preceding result was proved by Kōmura and Koshi [4] in the special case when $(E, C, \mathfrak{T})$ is a locally convex vector lattice.

A sequence (x_n) in an ordered convex space $(E, C, \mathfrak{T})$ is said to be *positive* if $x_n \in C$ for all n. Let $(E, C, \mathfrak{T})$ be a locally solid space and N the set of all natural numbers. We denote by $l^1[N, E]$ (resp. $l^1(N, E)$) the space consisting of all absolutely summable (resp. summable) sequences in E. It is not hard to see that if $(E, C, \mathfrak{J})$ is metrizable, then $l^1[N, E]$ is a solid subspace of the (ordered) product space (E^N, C^N).

Corollary 3. *A metrizable locally solid space* $(E, C, \mathfrak{J})$ *is nuclear if and only if it satisfies the following conditions*:

(a) $\mathfrak{J} = o(E, E')$;

(b) *each summable sequence in* E *is the difference of two positive summable sequences in* E.

Proof. Suppose that E is nuclear. Then $\mathfrak{J} = o(E, E')$ by Theorem 2, and $l^1(N, E) = l^1[N, E]$ in view of a well-known result. According to the remark preceding the corollary, $l^1[N, E]$ is a solid subspace of (E^N, C^N), hence (b) holds; therefore the condition is necessary. To see that it is also sufficient, we first note that each positive summable sequence in $(E, C, o(E, E'))$ is absolutely summable with respect to the topology $o(E, E')$. Therefore $l^1(N, E) = l^1[N, E]$. As E is metrizable, we conclude from Pietsch [8, (4.2.5)] that E is nuclear.

For a base normed space $(E, C, \|\cdot\|)$, the norm $\|\cdot\|$ is additive on C, it follows from Walsh [9, (3.2.5)] that the norm topology coincides with $o(E, E')$. But a normed space with infinite dimension is not nuclear, therefore condition (b) in Corollary 3 is essential.

3. **A generailzation of Andô-Ellis' theorem.** For a subset V of (E, C), we define

$$D(V) = \{x \in V\colon\ x = \lambda u - (1-\lambda)w,\ \lambda \in [0, 1] \text{ and } u, w \in V \cap C\}.$$

It is clear that if V is convex and circled, then $D(V) = \mathrm{co}\{-(V \cap C) \cup (V \cap C)\}$ and

$$(V \cap C - V \cap C)/2 \subset D(V) \subset V \cap C - V \cap C. \tag{4}$$

A subset V of (E, C) is said to be *decomposable* if $V = D(V)$. An ordered convex space $(E, C, \mathfrak{J})$ is called a *locally decomposable space* (C is *conormal* in $(E, \mathfrak{J})$ in the sense of Walsh [9]) if $\mathfrak{J}$ admits a neighbourhood base at 0 consisting of convex decomposable sets in E. According to formula (4), an ordered convex space $(E, C, \mathfrak{J})$ is locally decomposable if and only if C gives an open decomposition in $(E, \mathfrak{J})$ in the sense of Jameson [3, p. 94]. The topological dual E' of a locally decomposable space $(E, C, \mathfrak{J})$ is an order-convex subspace of E^*.

In this section $\beta(E', E)$ denotes the strong topology on E' while $\sigma(E', E)$ is the weak topology on E'.

Theorem 4. *Let* $(E, C, \mathfrak{J})$ *be an infrabarrelled ordered convex space with a countable fundamental system of* $\mathfrak{J}$*-bounded subsets of* E. *If* $(E, C, \mathfrak{J})$

is locally decomposable then $(E', C', \beta(E', E))$ *is locally o-convex.*

Proof. According to the hypotheses, $(E', C', \beta(E', E))$ is metrizable, therefore it is sufficient to show, by [10, (5.2)], that the order-convex hull (in E') of each $\beta(E', E)$-bounded subset B of E' is $\beta(E', E)$-bounded. Let B be such a set. B is $\mathfrak{J}$-equicontinuous, it follows from the polar characterization of local decomposability (see Walsh [9, (1.3.3)] or [10, (3.10)]) that $(B + C^*) \cap (B - C^*)$ is $\mathfrak{J}$-equicontinuous; consequently, the order-convex hull of B is $\beta(E', E)$-bounded.

Corollary 5. *Let* $(E, C, \mathfrak{J})$ *be an infrabarrelled ordered convex space with a countable fundamental system of bounded sets. If* $(E, C, \mathfrak{J})$ *is locally solid then so is* $(E', C', \beta(E', E))$.

Proof. It is known from [10, (6.3)] that an ordered convex space is locally solid if and only if it is both locally o-convex and locally decomposable Now Theorem 4 insures that $(E', C', \beta(E', E))$ is locally o-convex. On the other hand, by Schaefer's theorem [8, p. 220], the local o-convexity of $\mathfrak{J}$ implies that C' is a strict $\mathfrak{B}$-cone in $(E', \beta(E', E))$; by Jameson [3, (3.3.6)] (or [9, (1.2.4)]), $(E', C', \beta(E', E))$ is locally decomposable because $(E', \beta(E', E))$ is metrizable and surely bornological.

The following result should be compared with Jameson [3, (3.5.4)].

Theorem 6. *Let* $(E, C, \mathfrak{J})$ *be a metrizable ordered convex space such that* C *is* $\mathfrak{J}$*-complete. If* $(E', C', \beta(E', E))$ *is locally o-convex then* $(E, C, \mathfrak{J})$ *is locally decomposable, and hence* E *is* $\mathfrak{J}$*-complete.*

Proof. For any convex circled $\mathfrak{J}$-neighbourhood V of 0, it is known from [10, (2.11)] that $(D(V))^0 = (V^0 + C^*) \cap (V^0 - C^*) \cap E'$, therefore $(D(V))^0$ is the order-convex hull in E' of V^0. As E is metrizable and surely infrabarrelled, it follows from the local o-convexity of $(E', C', \beta(E', E))$ that $D(V)$ is a $\mathfrak{J}$-neighbourhood of 0. As C is complete, we conclude from the open mapping theorem (see Ng [5, Theorem 2]) that $(E, C, \mathfrak{J})$ is locally decomposable.

Finally, since $(E, C, \mathfrak{J})$ is locally decomposable for which C is $\mathfrak{J}$-complete, it follows from Klee's theoerm (see [10, (3.6)]) that E must be complete. Therefore the proof is complete.

The preceding theorem, together with a result of Schaefer [8, p. 220], shows that for a metrizable ordered convex space $(E, C, \mathfrak{J})$ such that C is $\mathfrak{J}$-complete, if $\beta(E', E)$ is locally o-convex then so is $\sigma(E', E)$.

Clearly an ordered normed space $(E, C, \|\cdot\|)$ is locally decomposable if and only if C is a strict $\mathfrak{B}$-cone in $(E, \|\cdot\|)$. Therefore Theorems 4 and 6 constitute a generalization of Andô-Ellis' theorem [1], [2] and [5, Theorem 3].

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