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ON THE RANGE OF A HYPONORMAL DERIVATION

JOSEPH G. STAMPFLI[1]

ABSTRACT. The inner derivation induced by a hyponormal operator has closed range if and only if the operator has finite spectrum.

Let $\mathcal{B}(\mathcal{H})$ denote the algebra of all bounded linear operators on a Hilbert space $\mathcal{H}$. Define the inner derivation

$$\Delta_A: \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{H}) \quad \text{by} \quad \Delta_A(X) = AX - XA$$

for $A, X \in \mathcal{B}(\mathcal{H})$. For $T \in \mathcal{B}(\mathcal{H})$ normal, Anderson and Foiaş [1] proved that the range of Δ_T (denoted by $\mathcal{R}(\Delta_T)$) is norm closed if and only if $\sigma(T)$ is finite. Their proof uses a number of deep results on decomposable operators and asymptotic commutativity. In this note we present a simple proof which enables us to extend their result to hyponormal operators.

The method of proof also permits us to answer partially a question raised by S. R. Caradus. To wit, when is $\mathcal{R}(\Delta_T) \cap \mathcal{K} = \Delta_T(\mathcal{K})$ where $\mathcal{K}$ is the ideal of compact operators. When T is hyponormal we show that equality holds if and only if $\sigma(T)$ is finite. The following result is a slight variation on a well-known result. See [3, Lemma 2] and subsequent material for example.

Lemma 1. *Let $T \in \mathcal{B}(\mathcal{H})$ be hyponormal. Let $\{\lambda_n\}_1^\infty$ be a sequence of distinct nonisolated boundary points of $\sigma(T)$. Let $\{\epsilon_n\}_1^\infty$ be a sequence of positive (nonzero) numbers converging to 0. Then there exists an orthonormal sequence $\{f_n\}_1^\infty$ of vectors from $\mathcal{H}$ such that*

(1) $\|(T - \lambda_n)f_n\| < \epsilon_n$ for $n = 1, 2, \ldots$, and

(2) $(f_j, Tf_n) = 0$ for $n = 1, \ldots, j-1$.

Theorem 1. *Let $T \in \mathcal{B}(\mathcal{H})$ be hyponormal. Then $\mathcal{R}(\Delta_T)$ is norm closed if and only if $\sigma(T)$ is finite.*

Proof. Let $\sigma(T)$ be infinite. Then $\sigma(T)$ has an infinite number of boundary points. We distinguish two cases. If $\sigma(T)$ has an infinite number of *isolated* boundary points $\{\lambda_n\}_1^\infty$, then by a well-known result [2] there exists an orthonormal sequence $\{f_n\}_1^\infty$ such that $Tf_n = \lambda_n f_n$ (this case is much easier to handle and the reader may wish to work it out first). If $\sigma(T)$ has an infinite

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number of distinct nonisolated boundary points $\{\lambda_n\}_1^\infty$, we can apply the previous lemma. In this case there exists an orthonormal sequence $\{f_n\}_1^\infty$ such that $\|(T-\lambda_n)f_n\| < \epsilon_n$ and $(f_j, Tf_n) = 0$ for $j > n$. We may further assume the λ_n's converge and we choose the ϵ_n's to satisfy the following conditions:

(1) $\epsilon_n > \epsilon_{n+1} > \cdots$;

(2) $\epsilon_n \leq |\lambda_{n+1} - \lambda_n|^2$ for $n = 1, 2, \ldots$;

(3) $\sum_{n=1}^\infty \epsilon_n \eta_n < \infty$ where $\eta_n = \max_{j=1,\ldots,n} |\lambda_{j+1} - \lambda_j|^{-1/2}$.

We set $\mathcal{H}_1 = \operatorname{clm}\{f_n\}_1^\infty$ and $\mathcal{H}_2 = \mathcal{H}_1^\perp$. If we write $Tf_n = \mu_n f_n + \delta_n$ where $(\delta_n, f_n) = 0$ then $|\mu_n - \lambda_n| < \epsilon_n$ and $\|\delta_n\| < \epsilon_n$ for $n = 1, 2, \cdots$. We will now define operators V_m such that $TV_m - V_mT$ will converge in norm to an operator $A \in \mathcal{B}(\mathcal{H})$, but $A \neq TW - WT$ for any $W \in \mathcal{B}(\mathcal{H})$. We define the unbounded operator V as follows: $Vf_n = |\lambda_{n+1} - \lambda_n|^{-1/2} f_{n+1}$ for $n = 1, 2, \cdots$ and $Vg = 0$ for $g \in \mathcal{H}_2$. Let P_m be the projection of $\mathcal{H}$ onto $\mathfrak{M}_m = \operatorname{clm}\{f_1, \ldots, f_m\}$ and set $V_m = VP_m$. We claim that $TV_m - V_mT$ converges in norm to an operator $A \in \mathcal{B}(\mathcal{H})$. Note first that

$$(TV_n - V_nT)f_j = \begin{cases} |\lambda_{j+1} - \lambda_j|^{-1/2}(\mu_{j+1} - \mu_j)f_{j+1} + |\lambda_{j+1} - \lambda_j|^{-1/2}\delta_{j+1} - V_n\delta_j \\ \hfill \text{for } j \leq n, \\ -V_n\delta_j \quad \text{for } j > n. \end{cases}$$

Thus

$$[\Delta_T(V_n) - \Delta_T(V_m)]f_j = \begin{cases} 0 \qquad \text{for } j \leq n \leq m, \\ -|\lambda_{j+1} - \lambda_j|^{-1/2}(\mu_{j+1} - \mu_j)f_{j+1} + |\lambda_{j+1} - \lambda_j|^{-1/2}\delta_{j+1} \\ \hfill + (V_m - V_n)\delta_j \quad \text{for } n < j \leq m, \\ (V_m - V_n)\delta_j \quad \text{for } n \leq m < j. \end{cases}$$

Note that $\|V_n\delta_j\| \leq \|V_n\| \, \|\delta_j\| \leq \eta_n \epsilon_j \leq \eta_j \epsilon_j$ for all n, j. (The last estimate follows by considering the two cases $n \leq j$, $n \geq j$ and using the fact that $(\delta_j, f_m) = 0$ for $m = j+1, j+2, \cdots$ in the latter.) Let $h \in \mathcal{H}_1$, and write $h = \sum_{j=1}^\infty a_j f_j$. By a standard argument we see that $\|[\Delta_T(V_n) - \Delta_T(V_m)]h\| \to 0$, uniformly in h as $n, m \to \infty$, since $|\lambda_{j+1} - \lambda_j|^{-1/2}|\mu_{j+1} - \mu_j| \to 0$ as $j \to \infty$ and $\sum \epsilon_j \eta_j < \infty$. We still must consider vectors $g \in \mathcal{H}_2$. For such a g,

$$(TV_n - V_nT)g = -V_nTg.$$

Let $Tg = \sum a_j f_j + w$ where $w \in \mathcal{H}_2$. Then $T^* f_j = \bar{\mu}_j f_j + \gamma_j$ where $(\gamma_j, f_j) = 0$. Since T is hyponormal $\|\gamma_j\| < \epsilon_j$. Thus $a_j = (Tg, f_j) = (g, T^* f_j) = (g, \gamma_j)$ and hence $|a_j| < \epsilon_j$ if g is a unit vector. Hence

$$(TV_n - V_nT)g = -\sum_{j=1}^n a_j |\lambda_{j+1} - \lambda_j|^{-1/2} f_{j+1}.$$

Finally

$$\|(\Delta_T(V_n) - \Delta_T(V_m))g\| \le \sum_{j=n}^{m} |a_j||\lambda_{j+1} - \lambda_j|^{-\frac{1}{2}} \le \sum_{n}^{m} \epsilon_j \eta_j$$

and the last term tends to zero as $n, m \to \infty$. Thus $\{\Delta_T(V_n)\}$ is a Cauchy sequence and hence it converges to an operator $A \in \mathcal{B}(\mathcal{H})$. To complete the first half of the proof we must show that $A \ne TW - WT$. Assume the contrary. Thus $((TW - WT)f_n, f_{n+1}) = (Af_n, f_{n+1})$ for all n and hence

$$(\mu_{n+1} - \mu_n)(Wf_n, f_{n+1}) + (Wf_n, \gamma_{n+1}) - (W\delta_n, f_{n+1})$$

$$= (Af_n, f_{n+1}) = (\mu_{n+1} - \mu_n)|\lambda_{n+1} - \lambda_n|^{-\frac{1}{2}}$$

since (δ_{n+1}, f_{n+1}) and $(V\delta_n, f_{n+1})$ are zero. Thus $|(Wf_n, f_{n+1})| \ge \frac{1}{2}|\lambda_{n+1} - \lambda_n|^{-\frac{1}{2}}$ for large n since $\epsilon_n/|\mu_{n+1} - \mu_n| \to 0$. This implies that W is unbounded, contrary to assumption. The other half of the proof will be sketched later.

We now turn to the question of Caradus mentioned in the introduction.

Theorem 2. *Let $T \in \mathcal{B}(\mathcal{H})$ be hyponormal. Then $\mathcal{R}(\Delta_T) \cap \mathcal{K} = \Delta_T(\mathcal{K})$ if and only if $\sigma(T)$ is finite.*

Proof. Again we prove only half the theorem now. Let $\sigma(T)$ be infinite. Proceed as in Theorem 1 and select λ_n, f_n and ϵ_n as before. This time however we define $Vf_n = f_{n+1}$ for $n = 1, 2, \cdots$ and $Vg = 0$ for $g \in \mathcal{H}_2$.

By estimates similar to those in Theorem 1, it is easy to see that $B = TV - VT$ is compact. (Indeed, the operator A in Theorem 1 is compact.) Note that $((TV - VT)f_n, f_{n+1}) = (\mu_{n+1} - \mu_n)$, since the other terms are zero. Assume that $B = TW - WT$ for some noncompact $W \in \mathcal{B}(\mathcal{H})$. Then

$$((TW - WT)f_n, f_{n+1}) = (\mu_{n+1} - \mu_n)(Wf_n, f_{n+1}) + (Wf_n, \gamma_{n+1}) - (W\delta_n, f_{n+1})$$

$$= (Bf_n, f_{n+1}) = (\mu_{n+1} - \mu_n).$$

Dividing the last equation by $(\mu_{n+1} - \mu_n)$ and letting $n \to \infty$ we see that $(Wf_n, f_{n+1}) \to 1$. Thus W is not compact and therefore $B \notin \Delta_T(\mathcal{K})$.

Remark. Let us now assume that T is hyponormal and $\sigma(T)$ is finite. In that case T must be normal. Thus we write $T = \Sigma_{j=1}^{n} \lambda_j E_j$ when the E_j's are just the spectral projections. For $V \in \mathcal{B}(\mathcal{H})$ write V as a matrix $[V_{ij}]$ on $\mathcal{H} = \Sigma_{j=1}^{n} \bigoplus E_j\mathcal{H}$. Then the ij entry in the matrix representation of $(TV - VT)$ is just $(\lambda_i - \lambda_j)V_{ij}$. This observation should make it clear that $\mathcal{R}(\Delta_T)$ is closed and moreover that $\mathcal{R}(\Delta_T) \cap \mathcal{K} = \Delta_T(\mathcal{K})$ since $[(\lambda_i - \lambda_j)V_{ij}]$ is compact if and only if V_{ij} is compact for all $i \ne j$.

Example. In the case of an arbitrary operator $T \in \mathcal{B}(\mathcal{H})$ we note that

$\sigma(T)$ finite does not imply $\mathcal{R}(\Delta_T) \cap \mathcal{K} = \Delta_T(\mathcal{K})$. For example let $T = \begin{vmatrix} 0 & Q \\ 0 & 0 \end{vmatrix}$ on $\mathcal{H} \oplus \mathcal{H}$ where Q is compact. If $R = \begin{vmatrix} A & B \\ C & D \end{vmatrix}$ then

$$TR - RT = \begin{vmatrix} QC & QD - AQ \\ 0 & -CQ \end{vmatrix}.$$

If we set $A = D = 0$ and $C = I$ then the operator $\begin{vmatrix} Q & 0 \\ 0 & -Q \end{vmatrix}$ is in $\mathcal{R}(\Delta_T) \cap \mathcal{K}$. But if Q is a selfadjoint compact operator with trivial kernel then $\begin{vmatrix} Q & 0 \\ 0 & -Q \end{vmatrix}$ is clearly not in $\Delta_T(\mathcal{K})$.

REFERENCES

1. Joel H. Anderson and C. Foiaş, *On the range of a derivation*, Pacific J. Math. (to appear).

2. Joseph G. Stampfli, *Hyponormal operators and spectral density*, Trans. Amer. Math. Soc. 117 (1965), 469–476; errata, ibid., 550. MR 30 #3375; 33 #4686.

3. ———, *Compact perturbations, normal eigenvalues and a problem of Salinas*, J. London Math. Soc. (2) 9 (1974), 165–175.

DEPARTMENT OF MATHEMATICS, INDIANA UNIVERSITY, BLOOMINGTON, INDIANA 47401