AN ALGORITHM FOR PARTITIONS

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ABSTRACT. A class of algorithms is described to represent the positive integers as sums of elements in a prescribed sequence of integers, and results are obtained on the densities of the integers that can be represented by such partitions with a bounded number of summands.

Let \( 1 = a_1 < a_2 < a_3 < \ldots \) be a strictly increasing sequence of positive integers, and let \( A = \{ a_i \} \). Every positive integer can be written in the form

\[
n = a_{i_1} + a_{i_2} + \cdots + a_{i_k},
\]

where the summands \( a_{i_j} \in A \) are chosen by the following "algorithm with \( p \) choices": If \( a_{i_1}, a_{i_2}, \ldots, a_{i_{k-1}} \) have been chosen, then \( a_{i_k} \) must be one of the \( p \) largest possible elements of \( A \), i.e. if \( a_{i_{k-1}} < n - (a_{i_1} + \cdots + a_{i_{k-1}}) \), then \( a_{i_k} \in \{ a_{i_1}, a_{i_2}, \ldots, a_{i_{k-1}} \} \). The number of summands in the shortest permissible partition (*) of \( n \) is called the \( p \)-length of \( n \), and denoted \( L_p(n) \). Let \( A^b_p \) be the set of all positive integers \( n \) such that \( L_p(n) \leq b \). Clearly, \( A^1_p = A \) for all \( p \), and \( A^b_p \subseteq A^{b+1}_p \) and \( A^b_p \subseteq A^{b+1}_p \) for all \( p \) and \( b \).

For example, let \( A = \{ i^2 \} \) be the sequence of squares. Then the "algorithm with 2 choices" gives four permissible partitions of 12, namely,

\[
9 + 1 + 1 + 1 = 4 + 4 + 4 = 4 + 4 + 1 + 1 + 1 = 4 + 1 + 1 + 1 + 1 + 1 + 1,
\]

and so \( L_2(12) = 3 \).

Katai [1] has studied the special case \( p = 1 \) when the representation (*) is unique.

Let \( S \) be any set of positive integers, and let \( S(N) \) denote the number of \( s \in S \) with \( s \leq N \). The lower asymptotic density of \( S \) is

\[
d(S) = \liminf_{N \to \infty} S(N)/N.
\]

The set \( S \) has density zero if \( \lim_{N \to \infty} S(N)/N = 0 \).

In this paper we study the densities of the sets \( A^b_p \). If \( A \) has density zero, then in Theorem 1 it is proved that all of the sets \( A^b_p \) have density zero. For example, if \( k \geq 2 \) and \( A \) is the sequence of \( k \)th powers, then Waring asserted and Hilbert proved that every positive integer is the sum of a bounded number of \( k \)th powers. But the sequence of \( k \)th powers has density zero.
zero, and so Theorem 1 shows that no algorithm with \( p \) choices is strong enough to settle Waring’s problem. In Theorem 2 a quantitative estimate is obtained for the size of \( A^h_p \) when \( A \) is a sequence distributed like the sequence of \( k \)th powers.

If \( A \) has positive lower asymptotic density, then \( A^h_p \) can be large. For example, if \( A = \{1, 2, \ldots, m - 1\} \cup \{m\} \), then \( A \) has density \( 1/m \) and \( A^h_p \) is the set of all positive integers for any \( p \geq 1 \) and \( h \geq 2 \). We prove in Theorem 3 that if \( d(A) > 0 \), then \( \lim_{h \to \infty} d(A^h_p) = 1 \) for all \( p \).

If \( f \) and \( \phi \) are functions of \( n \), then \( f = O(\phi) \) means that there exists a constant \( c \) such that \( |f(n)| < c\phi(n) \) for all \( n \).

**Theorem 1.** If \( A = \{a_i\} \) has density zero, then \( A^h_p \) has density zero for all \( p \) and \( h \).

**Proof.** Fix \( p \). For \( M \geq a_p \), choose \( j \) so that \( a_p \leq a_{j-1} \leq M < a_j \); that is, \( j - 1 = A(M) \). Let \( h \geq 2 \). If \( a_{j-1} < n \leq M \), then \( n \in A^h_p \) if and only if \( n - a_{j-1} \in A^h_{p-1} \) for some \( i = 1, 2, \ldots, p \). But \( n - a_{j-1} \leq M - a_{j-1} \), and so the number of such \( n \) with \( n - a_{j-1} \in A^h_{p-1} \) is not more than \( A^h_{p-1}(M - a_{j-1}) \). Therefore,

\[
A^h_p(M) - A^h_p(a_{j-1}) \leq \sum_{i=1}^{p} A^h_{p-1}(M - a_{j-1}) \leq pA^h_{p-1}(M - a_{j-1}).
\]

Applying this inequality with \( M = a_j - 1 \), we obtain

\[
A^h_p(a_j) - A^h_p(a_{j-1}) = 1 + A^h_p(a_{j-1}) - A^h_p(a_{j-1}) \leq 1 + pA^h_{p-1}(a_j - a_{j-1})
\]

Let \( N \geq a_p \). Then

\[
A^h_p(N) = A^h_p(N) - A^h_p(a_1(N)) + \sum_{j=p+1}^{A(N)} (A^h_p(a_j) - A^h_p(a_{j-1})) + A^h_p(a_1(N)) \leq pA^h_{p-1}(N - a_1(N)) + \sum_{j=p+1}^{A(N)} A^h_{p-1}(a_j - a_{j-1}) + A(N) + A^h_p(a_1(N)).
\]

Let the sequence \( A \) have density zero. Clearly, \( A^1_p = A \), and so \( A^1_p \) has density zero. Suppose that \( A^h_{p-1} \) has density zero for some \( h \geq 2 \). Choose \( \epsilon > 0 \). Then there exists a constant \( M_0 \) such that \( A^h_{p-1}(M) \leq M\varepsilon/2p^2 \) for all \( M \geq M_0 \). Therefore, \( A^h_{p-1}(M) \leq M_0 + M\varepsilon/2p^2 \) for all \( M \). Let \( N \geq a_p \). By inequality (**), we have
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\[ A^h_p(N) \leq p A^h_{p-1}(N - a(N) - \rho + 1) + p \sum_{j=\rho+1}^{A(N)} A^h_{p-1}(a_j - a_{j-p}) + A(N) + A^h_p(a_p) \]

\[ \leq p \left\{ M_0 \frac{(N - a(N) - \rho + 1) \epsilon}{2p^2} \right\} + p \sum_{j=\rho+1}^{A(N)} \left\{ M_0 \frac{(a_j - a_{j-p}) \epsilon}{2p^2} \right\} + A(N) + A^h_p(a_p) \]

\[ \leq p M_0 A(N) + p \left\{ \frac{A(N)}{N} + \sum_{j=A(N)-\rho+2}^{A(N)} \frac{a_j}{\rho^2} + A^h_p(a_p) + A(N) \right\} \]

\[ \leq p M_0 A(N) + N \epsilon/2 + A^h_p(a_p) + A(N). \]

Since \( A \) has density zero, there exists \( N_0 > a_p \) such that \( \lfloor (p M_0 + 1) A(N) + A^h_p(a_p) \rfloor < N \epsilon/2 \) for all \( N \geq N_0 \). Then \( A^h_p(N)/N < \epsilon \) for all \( N \geq N_0 \), and so \( A^h_p \) has density zero. The theorem follows by induction on \( h \).

**Theorem 2.** If \( A = \{a_i\} \) satisfies:

(i) \( A(N) = O(N^{1-\theta}) \) for \( 0 < \theta < 1 \),

(ii) \( a_i - a_{i-1} = O(i^\mu) \) for \( \mu > 1 \), and

(iii) \( (1 + \mu)(1 - \theta) \leq 1 \),

then \( A^h_p(N) = O(N^{1-\rho h}) \), where the implied constant depends on \( p \) and \( h \).

**Proof.** Let \( a_0 = 0 \). Then \( N \leq \sum_{i=1}^{A(N)+1} (a_i - a_{i-1}) \), and so, by (i) and (ii), we have \( N \leq c A(1-\theta)(1+\mu) \) for some constant \( c \). It follows that \( (1 - \theta)(1 + \mu) \geq 1 \), and so, by (iii), we have \( \mu = \theta/(1 - \theta) \). Clearly, \( A^1_p(N) = A(N) = O(N^{1-\theta}) \) and so the theorem holds for \( h = 1 \). Suppose that \( A^h_{p-1}(N) = O(N^{1-\rho h-1}) \) for some \( h \geq 2 \). If \( a_j - a_{j-1} = O(i^\mu) \), then \( a_j - a_{j-p} = O(i^\mu) \), where the implied constant depends on \( p \). By inequality (***) we have

\[ A^h_p(N) \leq p \sum_{j=\rho+1}^{A(N)+1} A^h_{p-1}(a_j - a_{j-p}) + A(N) + A^h_p(a_p) \]

\[ \leq p \sum_{j=\rho+1}^{A(N)+1} A^h_{p-1}(O(j^\mu)) + O(N^{1-\theta}) \]

\[ \leq \sum_{j=\rho+1}^{A(N)+1} O(j^{\mu(1-\rho h-1)}) + O(N^{1-\theta}) \]

\[ \leq A(N) O(A(N)^{\mu(1-\rho h-1)}) + O(N^{1-\theta}) \]

\[ < O(N^{1-\theta}) O(N^{(1-\theta)(1-\rho h-1)}) = O(N^{1-\rho h}). \]

The theorem follows by induction on \( h \).
Note that if \( A \) is the sequence of \( k \)th powers, then \( A(N) = O(N^{1/k}) = O(N^{1-(k-1)/k}) \) and \( i^k - (i-1)^k = O(i^{k-1}) \).

Theorem 3. If \( d(A) > 0 \), then \( \lim_{h \to \infty} d(A_p^h) = 1 \) for all \( p \).

Proof. Since \( d(A) > 0 \) and \( a_1 = 1 \), there exists \( \alpha > 0 \) such that \( A(N)/N > \alpha \) for all \( N \). In this case, Katai [1] has proved that for all \( p \) and \( N \),

\[
\alpha \sum_{n=1}^{N} L_p(n) \leq N.
\]

But also

\[
\alpha h(N - A_p^h(N)) \leq \alpha \sum_{n=1}^{N} L_p(n).
\]

Let \( \epsilon > 0 \), and choose \( h > 1/(\alpha \epsilon) \). Then

\[
A_p^h(N)/N \geq 1 - 1/\alpha h > 1 - \epsilon
\]

for all \( N \). Then \( d(A_p^h) \geq 1 - \epsilon \) for all \( h \geq 1/(\alpha \epsilon) \), and so \( \lim_{h \to \infty} d(A_p^h) = 1 \).

REFERENCE


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