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AN ALGORITHM FOR PARTITIONS

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ABSTRACT. A class of algorithms is described to represent the positive integers as sums of elements in a prescribed sequence of integers, and results are obtained on the densities of the integers that can be represented by such partitions with a bounded number of summands.

Let $1 = a_1 < a_2 < a_3 < \ldots$ be a strictly increasing sequence of positive integers, and let $A = \{a_i\}$. Every positive integer can be written in the form

$$(*) \qquad n = a_{i_1} + a_{i_2} + \cdots + a_{i_k},$$

where the summands $a_{i_j} \in A$ are chosen by the following "algorithm with p choices": If $a_{i_1}, a_{i_2}, \ldots, a_{i_{k-1}}$ have been chosen, then a_{i_k} must be one of the p largest possible elements of A, i.e. if $a_{s-1} \leq n - (a_{i_1} + \ldots + a_{i_{k-1}}) < a_s$, then $a_{i_k} \in \{a_{s-1}, a_{s-2}, \ldots, a_{s-p}\}$. The number of summands in the shortest permissible partition $(*)$ of n is called the *p-length of n*, and denoted $L_p(n)$. Let A_p^h be the set of all positive integers n such that $L_p(n) \leq h$. Clearly, $A_p^1 = A$ for all p, and $A_p^h \subset A_{p+1}^h$ and $A_p^h \subset A_p^{h+1}$ for all p and h.

For example, let $A = \{i^2\}$ be the sequence of squares. Then the "algorithm with 2 choices" gives four permissible partitions of 12, namely, $9 + 1 + 1 + 1 = 4 + 4 + 4 = 4 + 4 + 1 + 1 + 1 + 1 = 4 + 1 + 1 + 1 + 1 + 1 + 1 + 1 + 1$, and so $L_2(12) = 3$.

Katai [1] has studied the special case $p = 1$ when the representation $(*)$ is unique.

Let S be any set of positive integers, and let $S(N)$ denote the number of $s \in S$ with $s \leq N$. The *lower asymptotic density* of S is

$$d(S) = \liminf_{N \to \infty} S(N)/N.$$

The set S has *density zero* if $\lim_{N\to\infty} S(N)/N = 0$.

In this paper we study the densities of the sets A_p^h. If A has density zero, then in Theorem 1 it is proved that all of the sets A_p^h have density zero. For example, if $k \geq 2$ and A is the sequence of kth powers, then Waring asserted and Hilbert proved that every positive integer is the sum of a bounded number of kth powers. But the sequence of kth powers has density

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zero, and so Theorem 1 shows that no algorithm with p choices is strong enough to settle Waring's problem. In Theorem 2 a quantitative estimate is obtained for the size of A_p^h when A is a sequence distributed like the sequence of kth powers.

If A has positive lower asymptotic density, then A_p^h can be large. For example, if $A = \{1, 2, \ldots, m-1\} \cup \{im\}_{i=1}^{\infty}$, then A has density $1/m$ and A_p^h is the set of *all* positive integers for any $p \geq 1$ and $h \geq 2$. We prove in Theorem 3 that if $d(A) > 0$, then $\lim_{h\to\infty} d(A_p^h) = 1$ for all p.

If f and ϕ are functions of n, then $f = O(\phi)$ means that there exists a constant c such that $|f(n)| < c\phi(n)$ for all n.

Theorem 1. *If $A = \{a_i\}$ has density zero, then A_p^h has density zero for all p and h.*

Proof. Fix p. For $M \geq a_p$, choose j so that $a_p \leq a_{j-1} \leq M < a_j$; that is, $j - 1 = A(M)$. Let $h \geq 2$. If $a_{j-1} < n \leq M$, then $n \in A_p^h$ if and only if $n - a_{j-i} \in A_p^{h-1}$ for some $i = 1, 2, \ldots, p$. But $n - a_{j-i} \leq M - a_{j-i}$, and so the number of such n with $n - a_{j-i} \in A_p^{h-1}$ is not more than $A_p^{h-1}(M - a_{j-i})$. Therefore,

$$A_p^h(M) - A_p^h(a_{j-1}) \leq \sum_{i=1}^{p} A_p^{h-1}(M - a_{j-i}) \leq pA_p^{h-1}(M - a_{j-p}).$$

Applying this inequality with $M = a_j - 1$, we obtain

$$A_p^h(a_j) - A_p^h(a_{j-1})$$

$$= 1 + A_p^h(a_j - 1) - A_p^h(a_{j-1}) \leq 1 + pA_p^{h-1}(a_j - a_{j-p})$$

Let $N \geq a_p$. Then

$$A_p^h(N) = A_p^h(N) - A_p^h(a_{A(N)}) + \sum_{j=p+1}^{A(N)} \{A_p^h(a_j) - A_p^h(a_{j-1})\} + A_p^h(a_p)$$

$$(**)\qquad \leq pA_p^{h-1}(N - a_{A(N)-p+1}) + p\sum_{j=p+1}^{A(N)} A_p^{h-1}(a_j - a_{j-p}) + A(N) + A_p^h(a_p).$$

Let the sequence A have density zero. Clearly, $A_p^1 = A$, and so A_p^1 has density zero. Suppose that A_p^{h-1} has density zero for some $h \geq 2$. Choose $\epsilon > 0$. Then there exists a constant M_0 such that $A_p^{h-1}(M) \leq M\epsilon/2p^2$ for all $M \geq M_0$. If $M < M_0$, then $A_p^{h-1}(M) \leq M < M_0$. Therefore, $A_p^{h-1}(M) \leq M_0 + M\epsilon/2p^2$ for all M. Let $N \geq a_p$. By inequality (**), we have

$$A_p^h(N) \leq pA_p^{h-1}(N - a_{A(N)-p+1}) + p \sum_{j=p+1}^{A(N)} A_p^{h-1}(a_j - a_{j-p}) + A(N) + A_p^h(a_p)$$

$$\leq p\left\{M_0 + \frac{(N - a_{A(N)-p+1})\epsilon}{2p^2}\right\}$$

$$+ p \sum_{j=p+1}^{A(N)} \left\{M_0 + \frac{(a_j - a_{j-p})\epsilon}{2p^2}\right\} + A(N) + A_p^h(a_p)$$

$$\leq pM_0A(N) + p\left\{N + \sum_{j=A(N)-p+2}^{A(N)} a_j\right\}\epsilon\Big/2p^2 + A_p^h(a_p) + A(N)$$

$$\leq pM_0A(N) + N\epsilon/2 + A_p^h(a_p) + A(N).$$

Since A has density zero, there exists $N_0 > a_p$ such that $\{(pM_0 + 1)A(N) + A_p^h(a_p)\} < N\epsilon/2$ for all $N \geq N_0$. Then $A_p^h(N)/N < \epsilon$ for all $N \geq N_0$, and so A_p^h has density zero. The theorem follows by induction on h.

Theorem 2. *If* $A = \{a_i\}$ *satisfies*:

(i) $A(N) = O(N^{1-\theta})$ *for* $0 < \theta < 1$,

(ii) $a_i - a_{i-1} = O(i^{\mu})$ *for* $\mu > 1$, *and*

(iii) $(1 + \mu)(1 - \theta) \leq 1$,

then $A_p^h(N) = O(N^{1-\theta^h})$, *where the implied constant depends on* p *and* h.

Proof. Let $a_0 = 0$. Then $N \leq \Sigma_{i=1}^{A(N)+1}(a_i - a_{i-1})$, and so, by (i) and (ii), we have $N \leq cN^{(1-\theta)(1+\mu)}$ for some constant c. It follows that $(1 - \theta)(1 + \mu) \geq 1$, and so, by (iii), we have $\mu = \theta/(1 - \theta)$. Clearly, $A_p^1(N) = A(N) = O(N^{1-\theta})$ and so the theorem holds for $h = 1$. Suppose that $A_p^{h-1}(N) = O(N^{1-\theta^{h-1}})$ for some $h \geq 2$. If $a_i - a_{i-1} = O(i^{\mu})$, then $a_i - a_{i-p} = O(i^{\mu})$, where the implied constant depends on p. By inequality (**) we have

$$A_p^h(N) \leq p \sum_{j=p+1}^{A(N)+1} A_p^{h-1}(a_j - a_{j-p}) + A(N) + A_p^h(a_p)$$

$$\leq p \sum_{j=p+1}^{A(N)+1} A_p^{h-1}(O(j^{\mu})) + O(N^{1-\theta})$$

$$\leq \sum_{j=p+1}^{A(N)+1} O(j^{\mu(1-\theta^{h-1})}) + O(N^{1-\theta})$$

$$\leq A(N)O(A(N)^{\mu(1-\theta^{h-1})}) + O(N^{1-\theta})$$

$$\leq O(N^{1-\theta})O(N^{(1-\theta)\mu(1-\theta^{h-1})}) = O(N^{1-\theta^h}).$$

The theorem follows by induction on h.

Note that if A is the sequence of kth powers, then $A(N) = O(N^{1/k}) = O(N^{1-(k-1)/k})$ and $i^k - (i-1)^k = O(i^{k-1})$.

Theorem 3. *If* $d(A) > 0$, *then* $\lim_{h\to\infty} d(A_p^h) = 1$ *for all* p.

Proof. Since $d(A) > 0$ and $a_1 = 1$, there exists $\alpha > 0$ such that $A(N)/N > \alpha$ for all N. In this case, Katai [1] has proved that for all p and N,

$$\alpha \sum_{n=1}^{N} L_p(n) \leq N.$$

But also

$$\alpha h(N - A_p^h(N)) \leq \alpha \sum_{n=1}^{N} L_p(n).$$

Let $\epsilon > 0$, and choose $h > 1/(\alpha\epsilon)$. Then

$$A_p^h(N)/N \geq 1 - 1/\alpha h > 1 - \epsilon$$

for all N. Then $d(A_p^h) \geq 1 - \epsilon$ for all $h \geq 1/(\alpha\epsilon)$, and so $\lim_{h\to\infty} d(A_p^h) = 1$.

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