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THE SCHNIRELMANN DENSITY OF THE SUMS OF THREE SQUARES

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ABSTRACT. The number in the title is $5/6$, which is the same as the asymptotic density of the set. The best possible upper and lower bounds (of the form $5x/6 + A + B \log x$) are obtained for the number of positive integers less than x which are a sum of three squares.

For $x > 0$ let $S(x)$ and $N(x)$ be the numbers of positive integers not greater than x which are and are not a sum of three squares of integers, respectively. It is well known that $N(x)$ is the number of numbers $\leq x$ which are of the form $4^a(8b + 7)$ where a and b are nonnegative integers. Since there are $[(1 + x)/8]$ integers $k \leq x$ with $k \equiv 7 \pmod 8$, it is easy to see that

$$N(x) = \sum_{j=0}^{\infty} \left[\frac{1 + x4^{-j}}{8} \right].$$

Using $x - 1 < [x] \leq x$ and the fact that the infinite sum terminates at $j_0 = [\log_4(x/7)]$, one obtains the estimates [1, p. 250]

$$\frac{x}{6} - \frac{7 \log x}{8 \log 4} - 1 < N(x) < \frac{x}{6} + \frac{\log x}{8 \log 4},$$

which Landau [2] used to prove that the asymptotic density of the set of sums of three squares is $\lim_{x \to \infty} S(x)/x = 5/6$. It is possible *a priori* that either $S(x) \geq 5x/6$ for all x in which case the Schnirelmann density of this set is $\inf_x S(x)/x = 5/6$ or $S(x) < 5x/6$ for some x in which case the Schnirelmann density is less than $5/6$. The latter situation occurs for the set of squarefree numbers [4], which has asymptotic density $6/\pi^2$ but Schnirelmann density $53/88$. (The infimum is attained at $x = 176$.) We will show that for all positive integers x we have $S(x) \geq (5x + 1)/6$, so that the first case applies for sums of three squares.

Theorem 1.

$$N(x) \leq (x - 1)/6 \quad \textit{for } x \geq 1,$$

and equality holds if and only if $x = 2^n - 1$ *for some odd positive integer* n.

Proof. Clearly the relative maxima of $N(x) - x/6$ are at x of the form $x = 4^a(8b + 7)$. For such x with $a \geq 2$ we have

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$$N(x) = \sum_{j=0}^{a-1}\left[\frac{4^{a-j}(8b+7)}{8}\right] + \sum_{j=a}^{\infty}\left[\frac{1+4^{a-j}(8b+7)}{8}\right]$$

$$= \sum_{j=0}^{a-2}\frac{4^{a-j}(8b+7)}{8} + \left[\frac{4(8b+7)}{8}\right] + \sum_{j=0}^{\infty}\left[\frac{1+4^{-j}(8b+7)}{8}\right]$$

$$= \frac{4^a(8b+7)}{8}\sum_{j=0}^{a-2}4^{-j} + \left[4b+\frac{7}{2}\right] + N(8b+7)$$

$$= \frac{4^a(8b+7)}{8}\frac{4}{3}(1-4^{1-a}) + 4b + 3 + N(8b+7).$$

$$N(x) = x/6 - 4b/3 - 5/3 + N(8b+7). \tag{1}$$

A similar but simpler calculation shows that (1) holds also when $a = 1$.

We now prove by induction on b that

$$N(8b+7) \le (8b+7)/6 - 1/6, \tag{2}$$

and for all positive integers a that

$$N(4^a(8b+7)) \le (1/6)4^a(8b+7) - 2/3. \tag{3}$$

For fixed b (3) follows from (1) and (2):

$$N(4^a(8b+7)) = \frac{1}{6}4^a(8b+7) - \frac{4}{3}b - \frac{5}{3} + N(8b+7)$$

$$\le \frac{1}{6}4^a(8b+7) - \frac{4}{3}b - \frac{5}{3} + \frac{8b+7}{6} - \frac{1}{6}$$

$$= \frac{1}{6}4^a(8b+7) - \frac{2}{3}.$$

If $b = 0$, then $N(8b+7) = 1 = (8b+7)/6 - 1/6$. Let $b \ge 1$, $x = 8b+7$, and y be the largest integer $< x$ which is not a sum of three squares. Then $y = x - 8$, $x - 7$, or $x - 3$. If $y = x - 8$, then $N(y) \le y/6 - 1/6$, and

$$N(x) = N(y) + 1 \le y/6 - 1/6 + 1 = (x-8)/6 + 5/6$$

$$= x/6 - 3/6 < x/6 - 1/6.$$

If $y = x - 7$, then $N(y) \le y/6 - 2/3$, so

$$N(x) = N(y) + 1 \le y/6 - 2/3 + 1 = (x-7)/6 + 1/3$$

$$= x/6 - 5/6 < x/6 - 1/6.$$

If $y = x - 3$, then $N(y) \le y/6 - 2/3$, and

$$N(x) = N(y) + 1 \le y/6 - 2/3 + 1 = (x-3)/6 + 1/3 = x/6 - 1/6.$$

These inequalities show that $N(x) \le (x-1)/6$ for all $x \ge 1$ and that if equality holds, then $x = 1$ or $x = 7$ or for some positive integer b we have $x = 8b+7$ and $8b+4$ is not a sum of three squares. In the latter case the odd number $y = 2b+1$ cannot be a sum of three squares, whence $y = 8b' + 7$.

Then $x = 4y + 3$. Using (1) with $a = 1$ we find

$$(x-1)/6 - 1 = N(x) - 1 = N(x-3) = N(4y)$$
$$= (x-3)/6 - 4b'/3 - 5/3 + N(y)$$

so that $N(y) = (8b' + 7)/6 - 1/6 = (y-1)/6$. If b' is a positive integer, we can repeat the above steps with x replaced by y. Continuing this process we must eventually reach $y = 7$. Since $N(y) = (y-1)/6$ implies $N(4y+3) = (4y+3-1)/6$ by (1) again, we see that $N(x) = (x-1)/6$ if and only if x is one of the numbers a_n, where $a_0 = 1$ and $a_{n+1} = 4a_n + 3$. An easy induction shows that $a_n = 2^{2n+1} - 1$, and the theorem is proved.

Corollary. *For every positive integer x we have $S(x) \geq (5x+1)/6$.*

Proof. $S(x) = x - N(x)$ for every positive integer x.

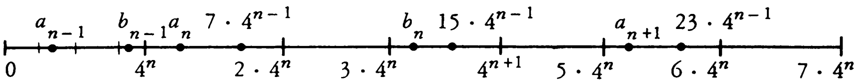
Corollary. *The Schnirelmann density of the set of all sums of three squares of integers is 5/6.*

We have reduced Landau's upper bound on $N(x)$ by about $(1/8)\log_4 x$. The next theorem narrows his lower bound on $N(x)$ by about $(5/24)\log_4 x$.

Theorem 2. *For all integers $x \geq 2$ we have*

$$N(x) \geq x/6 - (1/3)(1 + 2\log_4(1 + (3/4)(x-2))),$$

and equality holds if and only if x is of the form $x = 4(4^n - 1)/3 + 2$, where n is a nonnegative integer.



Proof. For each nonnegative integer n define positive integers a_n and b_n by $a_n = 4(4^n - 1)/3 + 2$, $b_n = 10(4^n - 1)/3 + 4$. Note that $a_0 = 2$, $b_0 = 4$, $a_{n+1} = b_n + 2 \cdot 4^n$, and $b_n = a_n + 2 \cdot 4^n$ for each nonnegative integer n, and that $a_n = a_{n-1} + 4^n$, $a_n \equiv b_n \equiv 6 \pmod 8$, and

$$4^n < a_n < 7 \cdot 4^{n-1} < 3 \cdot 4^n < b_n < 15 \cdot 4^{n-1}$$

for every positive integer n. Let $f(x) = x/6 - N(x)$. We will show first by induction on n that $f(a_n) = (2n+1)/3$ and $f(b_n) = (2n+2)/3$ for every nonnegative integer n. These equations are easily verified for small n, so we may assume $n \geq 2$. Every a_n and b_n is a sum of three squares. Let $\#A$ denote the cardinality of the set A. Since $a_n < 7 \cdot 4^{n-1}$, every number k between a_{n-1} and a_n which is not a sum of three squares is of the form $k = 4^a(8b+7)$ with $a \leq n-2$, and

$$\#\{a_{n-1} < k \leq a_n \mid k \text{ is not a sum of three squares}\}$$
$$= \#\{0 < k \leq 4^n \mid k \text{ is not a sum of three squares}\}$$
$$= 4^n\left(\frac{1}{8} + \frac{1}{8 \cdot 4} + \frac{1}{8 \cdot 16} + \cdots + \frac{1}{8 \cdot 4^{n-2}}\right) = \frac{4^n - 4}{6}.$$

Hence $N(a_n) = N(a_{n-1}) + (4^n - 4)/6$. Assume by induction that $f(a_{n-1}) = (2n-1)/3$. Then

$$f(a_n) = \frac{a_n}{6} - N(a_n) = \frac{a_n}{6} - N(a_{n-1}) - \frac{4^n}{6} + \frac{4}{6} = f(a_{n-1}) + \frac{2}{3} = \frac{2n+1}{3},$$

which is the first assertion. Every number k between a_n and b_n which is not a sum of three squares is of the form $k = 4^a(8b+7)$ with $a \leq n-1$. Hence

$$\begin{aligned} &\#\{a_n < k \leq b_n | k \text{ is not a sum of three squares}\} \\ &\quad = \#\{0 < k \leq 2 \cdot 4^n | k \text{ is not a sum of three squares}\} \\ &\quad = 1 + 2 \cdot (4^n - 4)/6. \end{aligned}$$

Therefore $N(b_n) = N(a_n) + 1 + (4^n - 4)/3$. We have already proved that $f(a_n) = (2n+1)/3$, so that

$$f(b_n) = \frac{b_n}{6} - N(b_n) = N(a_n) - \frac{2 \cdot 4^n}{6} - 1 + \frac{4}{3} = f(a_n) + \frac{1}{3} = \frac{2n+2}{3}.$$

Notice that $f(b_n) = f(a_n) + 1/3$ and $f(a_{n+1}) = f(b_n) + 1/3$. Also since every a_n and b_n is a sum of three squares we have

$$f(a_n - 1) = \frac{2n+1}{3} - \frac{1}{6} \quad \text{and} \quad f(b_n - 1) = \frac{2n+2}{3} - \frac{1}{6}.$$

Next we will prove that

$$x < b_n - 1 \quad \text{implies } f(x) \leq f(a_n) \tag{4}$$

and

$$x < a_{n+1} - 1 \quad \text{implies } f(x) \leq f(b_n). \tag{5}$$

We use induction on n and may assume $n \geq 2$. Suppose $x < b_n - 1$. Then $x - 2 \cdot 4^n < a_n - 1$. Let $y = x - t4^n$ ($t = 0, 1$, or 2) so that $0 < a_n - 4^n - 1 \leq y < a_n - 1$. We are assuming that $f(y) \leq f(b_{n-1})$. We have $f(x) = f(y) + t4^n/6 - m$, where

$$m = \#\{y < k \leq x | k \text{ is not a sum of three squares}\} = t(4^n - 4)/6 + z,$$

where

$$z = \begin{cases} 1 & \text{if } x \geq 7 \cdot 4^{n-1}, \\ 0 & \text{if } x < 7 \cdot 4^{n-1}. \end{cases}$$

Hence

$$f(x) = f(y) + 2t/3 - z \leq f(b_{n-1}) + 2t/3 - z \;= f(a_n) + (2t-1)/3 - z.$$

Now $z \geq 0$, so if $t = 0$, we have proved (4). Otherwise $t = 1$ or 2 so $x \geq a_n + 4^n > 7 \cdot 4^{n-1}$. Then z must be 1 and we have $f(x) \leq f(a_n) + (2t-1)/3 - 1 \leq f(a_n)$. This proves (4).

Now suppose $x < a_{n+1} - 1$. If $x \leq b_n$, we are done by the above. If $x > b_n$, let $y = x - 2 \cdot 4^n$. Then $a_n \leq y < b_n - 1$, so $f(y) \leq f(a_n)$ by induction, and $f(x) = f(y) + 2 \cdot 4^n/6 - m$, where

$$m = \#\{y < k \le x \mid k \text{ is not a sum of three squares}\} = 2(4^n - 4)/6 + 1,$$

since exactly one of the numbers $7 \cdot 4^{n-1}$, $15 \cdot 4^{n-1}$ is in the set. Therefore

$$f(x) = f(y) + 8/6 - 1 \le f(a_n) + 1/3 = f(b_n),$$

which is (5).

We now know that $f(x) \le g(x)$ for each positive integer x, where

$$g(x) = \begin{cases} (2n+1)/3 & \text{if } a_n \le x < b_n - 1, \\ (4n+3)/6 & \text{if } x = b_n - 1, \\ (2n+2)/3 & \text{if } b_n \le x < a_{n+1} - 1, \\ (4n+5)/6 & \text{if } x = a_{n+1} - 1. \end{cases}$$

Let $h(x) = (1/3)(1 + 2\log_4(1 + (3/4)(x-2)))$. Then $h(a_n) = g(a_n) = f(a_n)$ $(n \ge 0)$. To prove the theorem, it suffices to show that $h(x) > g(x)$ when $x \ge 2$ and x is not one of the a_n. For $x = a_n - 1$ we have $g(x) = (4n+1)/6$ and

$$h(x) = \frac{1}{3}(1 + 2n + 2\log_4(1 - 3/4^{n+1})) = g(x) + \frac{1}{6} + \frac{2}{3}\log_4(1 - 3/4^{n+1}).$$

But $1 + 4\log_4(1 - 3/s) > 0$ for $s > 6 + 3\sqrt{2}$ so that $h(a_n - 1) > g(a_n - 1)$ for positive integers n. (However for $n = 0$ we have $h(1) < g(1)$.) Now $b_n = (a_n + a_{n+1})/2$. Since $\log_4$ is concave down we have

$$h(b_n) > \frac{1}{2}(h(a_n) + h(a_{n+1})) = \frac{1}{2}(g(a_n) + g(a_{n+1})) = g(b_n).$$

Finally $b_n - 1 = ((a_n - 1) + (a_{n+1} - 1))/2$, whence $h(b_n - 1) > g(b_n - 1)$, and the proof is complete.

Note that the bounds on $f(x)$ given by Theorems 1 and 2 are the best possible of the form $A + B\log x$.

Corollary. *For all integers* $x \ge 2$ *we have*

$$\frac{5}{6}x + \frac{1}{6} \le S(x) \le \frac{5}{6}x + \frac{1}{3}\left(1 + 2\log_4\left(1 + \frac{3}{4}(x-2)\right)\right).$$

The set E of all positive integers which are not a sum of three squares has density $1/6$ but Schnirelmann density 0 because 1 is not in E. One might hope that if we adjoin some finite set of integers (including 1) to E, we would get a set with Schnirelmann density $1/6$. An easy calculation shows that the Schnirelmann density of $E \cup \{1\}$ is $3/22$ and that of $E \cup \{1, 2\}$ is $7/43$, both less than $1/6$. Indeed, if F is any finite set, then $E \cup F$ has Schnirelmann density $< 1/6$ because $h(x)$ is not bounded above.

The quadratic form $x^2 + y^2 + 2z^2$ represents all positive integers not of the form $4^a(16b + 14) = 2 \cdot 4^a(8b + 7)$. Let $S_1(x)$ and $N_1(x)$ be the number of positive integers not exceeding x which are and are not represented by this quadratic form. Then $N_1(x) = N(x/2)$ Theorem 1 gives $N_1(x) \le (x-2)/12$ for $x \ge 2$, where equality holds if and only if $x = 2^n - 2$ for some even positive integer n. Therefore

$$S_1(x) = x - N_1(x) \ge (11x+2)/12 \qquad (x \ge 2),$$

so the Schnirelmann density of the set of integers represented by $x^2 + y^2 + 2z^2$ is $11/12$, which is the same as the asymptotic density of the set.

It is well known [3] that $x^2 + 2y^2 + 2z^2$ and $x^2 + 2y^2 + 4z^2$ represent the same integers as $x^2 + y^2 + z^2$ and $x^2 + y^2 + 2z^2$, respectively, so we have analogous theorems for the Schnirelmann and asymptotic densities of the sets of integers represented by these quadratic forms.

However, it is not always the case that if a positive definite ternary quadratic form represents 1, then the Schnirelmann density of the set of integers it represents is the same as the asymptotic density. For example, let $S_2(x)$ and $N_2(x)$ be the numbers of positive integers $\le x$ which are and are not represented by the form $x^2 + y^2 + 3z^2$. This form represents all positive integers not of the form $9^a(9b+6)$. (See [3].) Hence

$$N_2(x) = \sum_{j=0}^{\infty} \left[\frac{3 + x9^{-j}}{9}\right],$$

and this sum actually terminates at $j_0 = [\log_9(x/6)]$. Let $f(n) = n/8 - N_2(n)$. The methods of Theorems 1 and 2 show that the local extrema of $f(n)$ are

$$f(3(9^n-1)/4) = -n/4 \qquad (n \ge 0)$$

and

$$f(5(9^n-1)/8) = 5n/8 \qquad (n \ge 0),$$

and lead to these upper and lower bounds on $N_2(x)$:

Theorem 3. *For every nonnegative integer x we have*

$$\frac{x}{8} - \frac{5}{8}\log_9\left(1 + \frac{8x}{5}\right) \le N_2(x) \le \frac{x}{8} + \frac{1}{4}\log_9\left(1 + \frac{4x}{3}\right),$$

where equality holds on the left if and only if x is of the form $x = 5(9^n-1)/8$, and equality holds on the right if and only if x is of the form $x = 3(9^n-1)/4$.

Here again the bounds on $f(x)$ are the best possible of the form $A + B\log x$.

The resultant inequalities on $S_2(x)$ make it easy to prove that the Schnirelmann density of the set of integers represented by the form $x^2 + y^2 + 3z^2$ is $5/6$ (infimum attained at $x = 6$), while the asymptotic density of the set is $7/8$.

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radiation in finite cavities given on p. 521 of H. P. Baltes and F. K. Kneubühl, Helv. Phys. Acta. 45 (1972), 481–529.

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