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A GENERALIZATION OF LUSIN'S THEOREM

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ABSTRACT. In this note we characterize σ-finite Riesz measures that allow one to approximate measurable functions by continuous functions in the sense of Lusin's theorem. We call such measures Lusin measures and show that not all σ-finite measures are Lusin measures.

It is shown that if a topological space X is either normal or countably paracompact, then every measure on X is a Lusin measure. A counterexample is given to show that these sufficient conditions are not necessary.

0. **Introduction and results.** We term a positive measure μ a *Riesz measure* if it has the properties stated in the Riesz representation theorem, i.e. if it is a positive measure on a locally compact Hausdorff space X with the additional properties:

(a) μ is defined on a σ-algebra M which contains all Borel sets in X.

(b) $\mu(K) < \infty$ for every compact set $K \subsetneq X$.

(c) For every $E \in M$, we have $\mu(E) = \inf\{\mu(V) | E \subset V, V \text{ is open}\}$.

(d) For every $E \in M$ such that either $\mu(E) < \infty$ or E is open we have $\mu(E) = \sup\{\mu(K) | K \subset E, K \text{ is compact}\}$.

Together, (a) and (c) state that μ is an outer regular Borel measure. Inner regularity is required by (d) only on open sets and sets of finite measure.

By a space, we will always mean a locally compact Hausdorff space, and by a measure, we will mean a Riesz measure.

The following is a standard result due to Lusin.

Lusin's theorem. *Suppose f is a complex measurable function on a space X, $\mu(X) < \infty$, and $\epsilon > 0$. Then there exists g, a continuous function on X, such that $\mu\{x | f(x) \neq g(x)\} < \epsilon$.*

We will call μ a *Lusin measure* if μ is a σ-finite Riesz measure on a locally compact Hausdorff space for which the conclusion of Lusin's theorem is true. A locally compact Hausdorff space X is a *Lusin space* if every σ-finite Riesz measure μ on X is a Lusin measure. In this language, Lusin's theorem says that a finite Riesz measure is a Lusin measure.

Lebesgue measure on $\mathbf{R}^n$ can easily be shown to be a Lusin measure. $\mathbf{R}^n$ is σ-compact and Lebesgue measure is σ-finite. Therefore it is natural to ask if all σ-compact topological spaces are Lusin spaces and if all σ-finite measures are Lusin measures. It is the purpose of this note to answer these and similar questions.

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We will show in §1 that not all σ-finite measures are Lusin measures. In fact, as we shall see, the real line with Lebesgue measure can be the support of a non-Lusin measure.

To the best of our knowledge, a simple characterization of Lusin measures has not occurred in the literature. We give such a characterization in

Theorem 1. *A σ-finite Riesz measure μ on a space X is a Lusin measure if and only if X has a countable, locally finite open cover $\{V_n\}$ with $\mu(V_n) < \infty$ for all n.*

We would like to thank E. Wilson for drawing our attention to the above condition which he observed was sufficient for μ to be a Lusin measure.

We shall prove Theorem 1 in §1. In §2 we discuss Lusin spaces. A characterization of Lusin spaces seems to be much more difficult than that of Lusin measures. However we will prove the following elementary sufficient condition.

Theorem 2. *Every locally compact space which is either normal or countably paracompact is a Lusin space.*

This answers our question about σ-compact spaces. Every σ-compact, locally compact Hausdorff space is normal, and hence is a Lusin space. The above sufficient conditions are not necessary. In §2 we shall exhibit a Lusin space that is neither normal nor countably paracompact.

If we restrict our attention to discrete measures, i.e. measures that are zero off a countable set, we can characterize Lusin spaces.

Theorem 3. *Let X be a locally compact space. Then every discrete σ-finite measure on X is a Lusin measure if and only if each pair of closed sets in X, one of which is countable and has no limit points, can be separated by disjoint open sets.*

1. **Lusin measures.** A collection of point sets is called *discrete* if the closures of these point sets are disjoint and any subcollection of these closures has a closed union. We denote the support of a continuous function g by supp(g).

Lemma. *Let $\{V_i\}$ be a countable cover of X by open sets of finite measure, and let $\epsilon > 0$. Then there exists a collection of compact sets $\{K_i\}$ and a collection of open sets $\{U_i\}$ such that $K_i \subset U_i \subset V_i$, $U_i \cap U_j = \emptyset$ for all $i \neq j$, and $\mu(X - \bigcup_{i=1}^{\infty} K_i) < \epsilon$. Moreover, the K_i can be chosen so that $\{K_i\}$ is discrete.*

Proof. Since μ is a Riesz measure, we can choose J_1 compact and U_1 open such that

$$J_1 \subset U_1 \subset \bar{U}_1 \subset V_1 \quad \text{and} \quad \mu(V_1 - J_1) < \epsilon/2^2.$$

Now given J_i and U_i for all $i < n$, choose J_n compact and U_n open such that

$$J_n \subset U_n \subset \bar{U}_n \subset V_n - \bigcup_{i<n} \bar{U}_i \quad \text{and} \quad \mu\left(V_n - \bigcup_{i<n} \bar{U}_i - J_n\right) < \frac{\epsilon}{2^{n+1}}.$$

Then

$$\mu\left(X - \bigcup_{i=1}^{\infty} J_i\right) \le \sum_{i=1}^{\infty} \mu\left(V_n - \bigcup_{i<n} \bar{U}_i - J_n\right) < \frac{\epsilon}{2},$$

and since $U_n \subset V_n - \bigcup_{i<n} \bar{U}_i$, $U_n \cap U_i = \emptyset$ for all $n \neq i$. Let O be an open set such that $O \supset (\bigcup_{i=1}^{\infty} U_i)^c$ and $\mu(O - (\bigcup_{i=1}^{\infty} U_i)^c) < \epsilon/2$. Define $K_i = J_i \cap O^c$. Then $\bigcup_{i=1}^{\infty} K_i$ is closed and $\{K_i\}$ and $\{U_i\}$ have the desired properties.

Proof of Theorem 1. Suppose μ is a Lusin measure on X. Then μ is σ-finite, so $X = \bigcup_{i=1}^{\infty} E_i$ where $\mu(E_i) < \infty$ for all i. Using the outer regularity of μ, choose V_i containing E_i so that $\mu(V_i - E_i) < 1$. The Lemma now applies to the cover $\{V_i\}$. Thus there exist collections $\{K_i\}$ and $\{U_i\}$ as in the Lemma, such that $\mu(X - \bigcup_{i=1}^{\infty} K_i) < 1$.

For $x \in K_i$ define $f(x) = i$. If x is not contained in any K_i, let $f(x) = 0$. f is a measurable function, so there exists a g, continuous on X, such that $\mu\{x | f(x) \neq g(x)\} < 1$. Then $\{g^{-1}((i, i+2)) | i \in \mathbf{Z}\}$ is a countable locally finite cover of sets of finite measure.

Conversely, suppose $\{V_i\}$ is a countable locally finite cover of X by open sets of finite measure. Let the measurable f and $\epsilon > 0$ be given. Choose $\{K_i\}$ and $\{U_i\}$ as in the Lemma such that $\mu(X - \bigcup_{i=1}^{\infty} K_i) < \epsilon/2$. By Lusin's theorem and the complete regularity of X, choose g_i, continuous on X, such that $\mu\{x \in K_i | g_i(x) \neq f(x)\} < \epsilon/2^{i+1}$ and $\operatorname{supp}(g_i) \subset U_i$.

Let $g(x) = \Sigma_{i=1}^{\infty} g_i(x)$. Then $\mu\{x | g(x) \neq f(x)\} < \epsilon$, and since $\{V_i\}$ is locally finite and $\operatorname{supp}(g_i) \subset U_i \subset V_i$, we have that $\Sigma_{i=1}^{\infty} g_i$ is a locally finite sum of continuous functions, so that g is continuous.

We now present an example of a σ-finite measure that is not a Lusin measure.

Let $\omega_1 + 1$ be the space of all ordinals less than or equal to the first uncountable ordinal ω_1 with the order topology. Let $\mathbf{R}^+ = [0, \infty]$ denote the extended real line with the usual topology. Give $X^* = (\omega_1 + 1) \times \mathbf{R}^+$ the product topology and let X denote the subspace $X^* - \{(\omega_1, \infty)\}$.

We now define a σ-algebra M and measure μ on X. Let $E \in M$ iff $E \cap (\{\omega_1\} \times \mathbf{R}^+)$ is of the form $\{\omega_1\} \times E'$ for some Lebesgue measurable subset $E' \subset [0, \infty)$. For $E \in M$ define $\mu(E) = m(E')$ where m denotes Lebesgue measure and E' is the set associated with E as above. Then μ is a σ-finite Riesz measure.

It is easy to prove that every continuous function of X into the real line is bounded [6, p. 122]. For $(\omega_1, x) \in \{\omega_1\} \times \mathbf{R}^+$ define $f((\omega_1, x)) = x$ and let f be zero elsewhere. Then clearly if g is continuous, $\mu\{x \mid f(x) \neq g(x)\} = \infty$, so that μ is not a Lusin measure.

2. **Lusin spaces.**

Proof of Theorem 2. Let μ be a σ-finite measure on X. Choose $\{K_i\}$ and $\{U_i\}$ as in the Lemma such that $\mu(X - \bigcup_{i=1}^{\infty} K_i) < 1$.

If X is countably paracompact, then the cover $\{U_i\} \cup \{(\bigcup_{i=1}^{\infty} K_i)^c\}$ can be refined to a locally finite cover of open sets of finite measure. If X is normal, then $\bigcup_{i=1}^{\infty} K_i$ can be separated from $(\bigcup_{i=1}^{\infty} U_i)^c$, so that again we can get such a refinement of $\{U_i\} \cup \{(\bigcup_{i=1}^{\infty} K_i)^c\}$. Thus X is a Lusin space if it is either countably paracompact or normal.

It should be pointed out that countable paracompactness and normality are very closely related [1], [4]. In fact, the question of whether every normal locally compact Hausdorff space is countably paracompact seems to be unanswered.

Proof of Theorem 3. Suppose every discrete σ-finite measure on X is a Lusin measure. Let C be a countable closed set with no limit points, and D be a closed set disjoint from C. The measure μ that assigns a unit mass to each point of C is a Riesz measure. Write $C = \{c_i \mid i \in \mathbf{Z}^+\}$ and define the measurable function $f(c_i) = i$ for all $c_i \in C$ and $f(x) = 0$ for all $x \notin C$. By assumption, there exists a continuous g such that $f(c_i) = g(c_i)$ for all $c_i \in C$. Let $U_i \subset g^{-1}((i - 1/3, i + 1/3))$ with $c_i \in U_i \subset \overline{U}_i \subset D^c$. Then since g is continuous $\mathrm{cl} \bigcup_{i=1}^{\infty} U_i = \bigcup_{i=1}^{\infty} \overline{U}_i \subset D^c$, and $\bigcup_{i=1}^{\infty} U_i$ is the desired open set separating C from D.

Now suppose that any two disjoint closed sets, one of which is countable and has no limit points, can be separated from each other with disjoint open sets. Let μ be a discrete measure on X. μ is σ-finite, so we can find $\{K_i\}$ and $\{U_i\}$ as in the Lemma such that $\mu\{X - \bigcup_{i=1}^{\infty} K_i\} < 1$. Since $\mu(K_i) < \infty$ and μ is discrete, there exists a $J_i \subset K_i$ such that J_i is a finite point set and $\mu(K_i - J_i) < 1/2^i$.

By assumption, we can separate $\bigcup_{i=1}^{\infty} J_i$ from $(\bigcup_{i=1}^{\infty} U_i)^c$ with an open set O, whose closure lies in $\bigcup_{i=1}^{\infty} U_i$. Let $V_i = U_i \cap O$. Then $\bigcup_{i=1}^{\infty} \overline{V}_i = \mathrm{cl} \bigcup_{i=1}^{\infty} V_i$, so that $\{V_i\} \cup \{(\bigcup_{i=1}^{\infty} J_i)^c\}$ is a countable, locally finite cover of open sets of finite measure. Thus μ is a Lusin measure.

In the proof of Theorem 2 we used the fact that X is a Lusin space if there exists a locally finite refinement of the cover $\{U_i\} \cup \{(\bigcup_{i=1}^{\infty} K_i)^c\}$, where $\{U_i\}$ and $\{K_i\}$ are as in the Lemma. The existence of such a refinement for each such cover is not a necessary condition on Lusin spaces. We shall exhibit a consistent example, discovered by M. Starbird and the author, of a Lusin space that does not have the above property and hence is not normal or countably paracompact.

Let L denote a compact Souslin line. We recall that this means

(1) L is a totally ordered set with first and last elements,

(2) L has the interval topology induced by the total ordering and L is compact and connected,

(3) each collection of disjoint intervals in L is countable, and finally,

(4) no open subset of L is separable.

There exists a dense subset D of L with the property that if $A \subset L$ is countable, then $\bar{A} \cap D$ is countable [5]. Lexicographically order the Cartesian product $S = L \times [0, 1]$, and endow S with the interval topology induced by this ordering. Let $X^* = (\omega + 1) \times S$ with the product topology, where $\omega + 1$ denotes the space of all ordinals less than or equal to the first infinite ordinal ω with the interval topology. Let $B = D \times (0, 1) \subset S$. Finally, denote by X the subspace $X^* - \{\{\omega\} \times B^c\}$.

We need some notation. Let S_n denote the square $\{n\} \times S$ and S_ω be the limit square $(\{\omega\} \times S) \cap X$. Denote $\{n\} \times L \times \{0, 1\}$, i.e. the Souslin edges of the square S_n, by J_n. Note that each J_n is compact.

Now $X - \bigcup_{i=1}^{\infty} J_n$ is an open subset of X that is normal and hence a Lusin space. Similarly S_ω^c is an open subset of X that is a Lusin space. Also, if $C \subset S_\omega$ is of the form $\bigcup_{i=1}^{\infty} K_i$, where $\{K_i\}$ is discrete, then C is Lindelöf, so C can be separated from the closed set $\bigcup_{i=1}^{\infty} J_n$. This is true because in a regular Hausdorff spcae any two disjoint closed Lindelöf subsets have disjoint neighborhoods. All that remains in showing that X is a Lusin space is the proof of the

Claim. Suppose $C = \bigcup_{i=1}^{\infty} K_i \subset X$ where $\{K_i\}$ is discrete, $C \subset \bigcup_{i=1}^{\infty} J_i$, and each K_i is the support of some measure. Then C and S_ω can be separated with disjoint open sets.

Proof. It is easy to show [2] that any subset of the Souslin line that is the support of a Riesz measure is separable. Therefore, $\{K_i\}$ is a countable collection of separable subsets, and we can find a separable closed set H contained in L such that $C \subset \omega \times H \times \{0, 1\}$. By property (3), every closed subset of the Souslin line is a G_δ. Therefore, there exists a collection $\{G_i'\}$ of open subsets of L, such that $H = \bigcap_{i=1}^{\infty} G_i'$. Let $G_i = \{i\} \times G_i' \times [0, 1] \subset X$. The open set $G = \bigcup_{i=1}^{\infty} G_i$ contains C, and $\bar{G}$ contains points in at most a countable number of the components of S_ω. Thus $\bar{G} \cap S_\omega$ is Lindelöf and can be separated from C with an open set U such that $C \subset U \subset \bar{U} \subset (\bar{G} \cap S_\omega)^c$. Now $C \subset (G \cap U) \subset \overline{G \cap U} \subset S_\omega^c$ and we have proven our claim.

Let μ be a measure on X and $\{K_i'\}$ and $\{U_i\}$ be as in the Lemma with $\mu(X - \bigcup_{i=1}^{\infty} K_i') < 1$. Let K_i denote the intersection of K_i' with the support of μ. Then the above shows that $\{U_i\} \cup \{\bigcup_{i=1}^{\infty} K_i\}$ can be refined to a locally finite cover, and by Theorem 1, μ is a Lusin measure. We conclude that X is a Lusin space.

To show that X is neither normal nor countably paracompact, it is enough to show that $\bigcup_{i=1}^{\infty} J_i$ cannot be separated from S_ω by disjoint open sets. But if O is an open set containing J_i, then O must contain all but a separable subset of the square S_i. Thus any open set that contains $\bigcup_{i=1}^{\infty} J_i$ must have a limit point in S_ω.

Since this example relies on the existence of the Souslin line, its construction requires the use of set theoretic conditions beyond the axiom of choice. It would be interesting to have an example of a nonnormal noncountably paracompact Lusin space that is free of such set theoretic considerations.

We have not given necessary and sufficient topological conditions on Lusin spaces. However, the above proof that X is a Lusin space suggests that the following condition may be necessary and sufficient:

(*) Any two disjoint closed sets, one of which is of the form $\bigcup_{i=1}^{\infty} K_i$ where $\{K_i\}$ is discrete and each K_i is a compact set that is the support of some measure, have disjoint neighborhoods.

Since it is known exactly which compact Hausdorff spaces can be the support of a measure [3], (*) can easily be translated into a purely topological condition.

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