ON THE INFINITE DIMENSIONALITY OF THE DOLBEAULT COHOMOLOGY GROUPS

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ABSTRACT. Let $M$ be an open subset of a Stein manifold without isolated points. Let $\mathcal{O}^p$ be the sheaf of germs of holomorphic $p$-forms on $M$. Then $H^q(M, \mathcal{O}^p)$ is either 0 or else infinite dimensional. $H^q(M, \mathcal{S})$ may be nonzero and finite dimensional if $M$ is the regular points of a Stein space or if $\mathcal{S}$ is an arbitrary coherent sheaf over $M$.

Let $M$ be a complex manifold. Let $\Omega^p$ be the sheaf of germs of holomorphic $p$-forms over $M$. If $M$ is a Stein manifold, then $H^q(M, \Omega^p) = 0$ for $q > 1$ [3, VIII.A, 14, Cartan's Theorem B, p. 243] while $H^0(M, \Omega^p)$ is an infinite dimensional Fréchet space, so long as $M$ does not consist of a finite number of points. If $M$ is a compact manifold, then $H^q(M, \Omega^p)$ is finite dimensional for all $p$ and all $q$ [3, VIII.A, 10, p. 245]. In this paper, we shall examine the possible dimensions for $H^q(M, \Omega^p)$, the Dolbeault cohomology groups, under the assumption that $M$ is an open subset of a Stein manifold and that $M$ does not have any isolated points. By considering the natural topology on $H^q(M, \Omega^p)$ Siu showed [7, Theorem A, p. 17], under much more general assumptions, that $H^q(M, \Omega^p)$ cannot be countably infinite dimensional. The topology on $H^q(M, \Omega^p)$ is that induced from the topology of uniform convergence on compact sets for all derivatives for $C^\infty$ differential forms. $H^q(M, \Omega^p)$ is then a linear topological space. Let $R^p,q$ be the closure of 0 in $H^q(M, \Omega^p)$. Then $H^q(M, \Omega^p)/R^p,q$ is a separable Fréchet space. Siu essentially showed that $R^p,q$ cannot be countably infinite dimensional. The main result of this paper is that $H^q(M, \Omega^p)/R^p,q$ is either 0 or infinite dimensional. The author previously proved a special case in [5, Theorem 4.5, p. 431]. Some examples are given which show that some special assumptions about $M$ and about the sheaf are needed.

Let $E^{p,q}$ be the $C^\infty$ differential forms on $M$ of type $(p, q)$. If $f$ is a holomorphic function on $M$, then $f$ operates on $E^{p,q}$ via multiplication. We shall also denote this endomorphism of $E^{p,q}$ by $f$. Let $\lambda$ be a holomorphic vector field on $M$, i.e., a section of the dual sheaf to $\Omega^1$. Then $\lambda$ induces a map, also denoted by $\lambda, \lambda: E^{p,q} \rightarrow E^{p-1,q}$ given by contraction, thinking of...
the vector field $\lambda$ and elements of $E^{p,q}$ and $E^{p-1,\overline{q}}$ as tensors. In local coordinates, if
\[
\lambda = \sum \lambda_k(z) \frac{\partial}{\partial z_k}, \quad 1 \leq k \leq n,
\]
and $\omega \in E^{p,q}$ is given by
\[
\omega = \sum \omega_{I,J}(z) dz^I \wedge d\overline{z}^J,
\]
where the summation is over the multi-indices $I = (i_1, \ldots, i_p)$, $i_1 < \cdots < i_p$, and $J = (j_1, \ldots, j_q)$, $j_1 < \cdots < j_q$, then
\[
\lambda(\omega) = \sum (-1)^{\epsilon+1} \lambda_k(z) \omega_{I,J}(z) dz^{I'} \wedge d\overline{z}^{J'}
\]
where the summation is over $1 \leq k \leq n$, $I = (j_1, \ldots, j_q)$ and $J = (i_1, \ldots, i_p)$ such that $k \in I$. $I'$ is obtained from $I$ by deleting $k$. $\epsilon$ is given by $l \ni k$, $k = i_\epsilon$. The map $\lambda$ commutes with the map $f$.

Let $D: E^{p,q} \rightarrow E^{p,q}$ be given by $D = \partial \circ \lambda + \lambda \circ \partial$. The following lemma is true for any complex manifold. $\partial f$ is a holomorphic 1-form and $\lambda(\partial f)$ is the function obtained by the usual operation of a vector field $\lambda$ on a function $f$.

Lemma. Let $D = \partial \circ \lambda + \lambda \circ \partial$. Then $D \circ f = f \circ D = \lambda(\partial f)$.

Proof. Let $\omega \in E^{p,q}$.

\[
D \circ f - f \circ D = \partial f \wedge \lambda(\omega) + \lambda(\partial f \wedge \omega).
\]

The right side of (3) is $C^\infty$-linear in $\omega$ and $C^\infty$-linear in $\lambda$. So to complete the verification of the Lemma, it suffices to evaluate the right side of (3) in local coordinates with $\lambda = \partial/\partial z_l$ and $\omega = dz^I \wedge d\overline{z}^J$. There are two cases, $l \notin l$ and $1 = i_1 \in l = (i_1, \ldots, i_p)$.

For $l \notin l$: $\lambda(\omega) = 0$ and $\lambda(\partial f \wedge \omega) = \partial f / \partial z_l \wedge \omega$, as needed.

For $1 \in l$: $\lambda(\omega) = dz^{I'} \wedge d\overline{z}^J$ where $I' = (i_2, \ldots, i_p)$.

\[
\partial f \wedge \lambda(\omega) = \sum \frac{\partial f}{\partial z_k} dz^k \wedge dz^{I'} \wedge d\overline{z}^J, \quad k \notin I',
\]
\[
\partial f \wedge \omega = \sum \frac{\partial f}{\partial z_k} dz^k \wedge dz^I \wedge d\overline{z}^J, \quad k \notin l.
\]
\begin{align}
\lambda(\partial f \wedge \omega) &= \sum (-1)^k \frac{\partial f}{\partial z_{k}} \, dz^k \wedge dz^i \wedge \overline{dz}^j, \quad k \neq l.
\end{align}

Theorem. Let \( M \) be an open subset of a Stein manifold having no isolated points. Then, for any \( p \) and \( q \), \( H^q(M, \Omega^p) \) is either 0 or else infinite dimensional. Let \( R^{p,q} \) be the closure of 0 in \( H^q(M, \Omega^p) \). Then \( H^q(M, \Omega^p)/R^{p,q} \) is either 0 or else infinite dimensional.

Proof. The proof for \( H^q(M, \Omega^p) \) is exactly like that for \( H^q(M, \Omega^p)/R^{p,q} \), leaving out topological considerations; therefore we omit it.

For the sake of notational simplicity, let \( H = H^q(M, \Omega^p)/R^{p,q} \). We shall show that if \( H \) is finite dimensional, then \( H = 0 \). For \( f \) holomorphic on \( M \), the action of \( f \) on \( E^{p,q} \) is continuous and commutes with \( \overline{\partial} \). Thus, \( f \) induces a continuous endomorphism on \( H^q(M, \Omega^p) \) and on \( H \).

Without loss of generality, we may assume that \( M \) is a subset of a connected Stein manifold \( S \) of dimension \( n > 0 \). Let \( I \) be the set of holomorphic functions on \( S \) which, after restriction to \( M \), induce the zero map of \( H \) to itself. It suffices to show that \( 1 \in I \). Let \( z_1, \ldots, z_{2n+1} \) be holomorphic functions on \( S \) which separate points [3, Theorem VII.C.10, p. 224]. Each \( z_i \) acts on \( H \) and so has a minimal polynomial \( p_i(z_i) \), under the finite dimensional assumption on \( H \). \( p_i(z_i) \in I \). \( p_i(z_i) \) has only a finite number of roots in \( z_i \). So the common zeros on \( S \) for \( p_1(z_1), \ldots, p_{2n+1}(z_{2n+1}) \) consist of only a finite number of points, say \( P_1, \ldots, P_N \), in \( S \).

\( \partial : E^{p,q} \to E^{p+1,q} \) is continuous and anticommutes with \( \overline{\partial} \). If \( \lambda \) is a holomorphic vector field on \( S \), then \( \lambda : E^{p,q} \to E^{p-1,q} \) is continuous and commutes with \( \overline{\partial} \). So \( \lambda \circ \partial \) and \( \partial \circ \lambda \) both induce endomorphisms of \( H \). The Lemma also holds for the induced maps on \( H \). Thus, if \( f \in I \), then also \( \lambda(\partial f) \in I \). Consider a \( P_j \in S \) from above. Let \( f \in I \) have a zero at \( P_j \) of minimal total order. \( f \neq 0 \) since \( p_i(z_i) \in I \). We claim that \( f(P_j) \neq 0 \), for suppose otherwise. Since \( S \) is a Stein manifold of positive dimension, by an application of Cartan's Theorem B, we can specify the tangent vector at \( P_j \) for a vector field \( \lambda \) on \( S \). For a suitable choice for \( \lambda \), \( \lambda(\partial f) \) will have a zero of lower total order at \( P_j \) than has \( f \). So for each \( P_j \), there exists an \( f_j \in I \) such that \( f_j(P_j) \neq 0 \). \( p_1(z_1), \ldots, p_{2n+1}(z_{2n+1}) \), \( f_1, \ldots, f_N \) are then elements of \( I \) with no common zeroes. By [3, Corollary VIII.A.16, p. 244] there exist holomorphic functions \( \{g_k\} \) on \( S \) such that \( \sum g_i p_i(z_i) + \sum g_j f_j = 1 \). Thus \( 1 \in I \) and \( H = 0 \), as desired.

The Theorem does not hold under the weaker assumption that \( M \) is an open subset of a Stein space, even if \( M \) itself is a manifold. Consider, for example, a Riemann surface \( R \) of genus at least 1 embedded as the 0-section
of a negative vector bundle $V$ of rank 4. See [2]. $V$ can be taken to be the direct sum of 4 line bundles, each of negative Chern class. Let $\mathcal{O} = \Omega^0$ be the sheaf of germs of holomorphic functions. Then $H^1(V, \mathcal{O}) \neq 0$ since $H^1(V, \mathcal{O})$ may be expanded in a power series along the fibers [2, pp. 343–344]. Also, $H^1(V, \mathcal{O})$ is finite dimensional since $V$ is strictly pseudoconvex [1, Theorem 11, p. 239]. Let $M = V - R$. Then the restriction map induces an isomorphism $H^1(V, \mathcal{O}) \cong H^1(M, \mathcal{O})$ by [6, Corollary, p. 351]. Thus $H^1(M, \mathcal{O})$ is nonzero and finite dimensional. By blowing down $R$ to a point $p$, we obtain a Stein space $S$ with $M$ the complement of the singular point $p$.

This example also shows that $\Omega^p$ in the Theorem cannot be replaced by an arbitrary coherent sheaf. Namely, near $p$, $S$ may be embedded as a subvariety $X$ of a polydisc $\Delta$.

\[ H^1(S - p, \mathcal{O}) \cong H^1(X - p, \mathcal{O}) \text{ by [4, Theorem 2.2, p. 105].} \]

$X^\mathcal{O}$ is a coherent sheaf on $\Delta$, but $H^1(\Delta - p, X^\mathcal{O})$ is nonzero and finite dimensional.

BIBLIOGRAPHY