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AN EQUIVALENCE THEOREM ON GENERATING FUNCTIONS[1]

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ABSTRACT. The present paper establishes an equivalence theorem on generating relations for a sequence of functions $\{f_n(z)\}$ defined by the Rodrigues formula (1) below. It is also shown how this theorem may be applied to a fairly large variety of special functions including, for instance, the classical orthogonal polynomials.

1. **The main result.** We prove the following generalization of certain results on generating functions due to F. Brafman [1]:

Theorem. *Let the function $F(z)$ be holomorphic on a domain D of the complex z-plane, and define*

$$f_n(z) = (n!)^{-1} D_z^n\{(az+b)^n F(z)\}, \qquad D_z = d/dz, \tag{1}$$

for $n = 0, 1, 2, \ldots$, where a and b are complex constants, not both zero. Also let $\{\lambda_n\}$ be any sequence of complex numbers for which

$$1/R = \limsup_{n\to\infty} |\lambda_n|^{1/n} \tag{2}$$

is finite, or for which

$$R = \lim_{n\to\infty} |\lambda_n/\lambda_{n+1}| \tag{3}$$

exists and is positive. Suppose further that

$$A_n = \sum_{k=0}^{[n/N]} \binom{n}{Nk} \lambda_k w^k, \qquad n = 0, 1, 2, \ldots, \tag{4}$$

for some positive integer N and some complex number w.

Then

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$$\sum_{n=0}^{\infty} A_n f_n(z) t^n = \frac{1}{1-at} \sum_{k=0}^{\infty} \lambda_k f_{Nk}\left(\frac{z+bt}{1-at}\right)\left[\frac{wt^N}{(1-at)^N}\right]^k, \tag{5}$$

for some domain in the complex t-plane that includes the origin.

Proof. By the Cauchy-Hadamard theorem, R in (2) is the radius of convergence of the series

$$\sum_{n=0}^{\infty} \lambda_n Z^n. \tag{6}$$

By the d'Alembert ratio test, R is also given by (3) in case that limit exists.

Note that

$$\left|\frac{t}{1-at}\right| < \rho \quad \text{whenever } |t| < T(\rho) = \frac{\rho}{\rho|a|+1}. \tag{7}$$

Let z be any point in D, and let C be a circle of radius r (with centre at z) lying entirely within D. If we define

$$z' = (z+bt)/(1-at), \tag{8}$$

so that

$$z' - z = (az+b)t/(1-at),$$

then, in view of (7), z' is within a distance $r/2$ of z whenever

$$|t| < T(r/2|az+b|). \tag{9}$$

By definition (1) and Cauchy's integral formula, we have

$$f_m(z') = \frac{1}{2\pi i}\int_C F(\zeta)\frac{(a\zeta+b)^m}{(\zeta-z')^{m+1}}\,d\zeta, \qquad m = 0, 1, 2, \ldots, \tag{10}$$

whence the right-hand side of (5) may be written

$$\Omega = \frac{1}{1-at}\sum_{k=0}^{\infty} \lambda_k \frac{1}{2\pi i}\int_C F(\zeta)\frac{(a\zeta+b)^{Nk}}{(\zeta-z')^{Nk+1}}\,d\zeta\left(\frac{wt^N}{(1-at)^N}\right)^k. \tag{11}$$

Now consider the series

$$S(\zeta) = \sum_{k=0}^{\infty} \lambda_k w^k \left(\frac{a\zeta+b}{\zeta-z'}\frac{t}{1-at}\right)^{Nk} \tag{12}$$

when ζ is on C. Since $|z'-z| < r/2$, we have $|\zeta - z'| > r/2$. Suppose $|a\zeta+b| < K$ when ζ is on C. Suppose further that

$$|t| < T\left(\left[\frac{R}{|w|}\right]^{1/N}\frac{r}{2K}\right). \tag{13}$$

Then, by (7),

$$\left|\frac{t}{1-at}\right| < \left[\frac{R}{|w|}\right]^{1/N}\frac{r}{2K}.$$

Therefore

$$\left|\frac{a\zeta+b}{\zeta-z'}\frac{t}{1-at}\right| < \left[\frac{R}{|w|}\right]^{1/N}$$

It follows that series (12) converges uniformly in ζ when (3) holds.

This evidently justifies the inversion of the order of summation and integration in (11), and we thus have

$$\Omega = \frac{1}{1-at}\frac{1}{2\pi i}\int_C F(\zeta)S(\zeta)\frac{d\zeta}{\zeta-z'}. \tag{14}$$

By (8) and (12), for ζ on C, we get

$$\Delta = \frac{S(\zeta)}{(1-at)(\zeta-z')} = \frac{1}{\zeta-z}\sum_{k=0}^{\infty}\lambda_k w^k\frac{W^{Nk}}{(1-W)^{Nk+1}}, \tag{15}$$

where, for convenience,

$$W = (a\zeta+b)t/(\zeta-z). \tag{16}$$

If

$$|t| < r/K, \tag{17}$$

then $|W| < 1$, and the denominator in series (15) can be expanded by means of the binomial theorem, giving

$$\begin{aligned}\Delta &= \frac{1}{\zeta-z}\sum_{k=0}^{\infty}\lambda_k w^k\sum_{m=0}^{\infty}\binom{m+Nk}{Nk}W^{m+Nk}\\ &= \frac{1}{\zeta-z}\sum_{k=0}^{\infty}\lambda_k w^k\sum_{n=Nk}^{\infty}\binom{n}{Nk}W^n.\end{aligned} \tag{18}$$

To justify the interchange of the order of summation in this double series, we require that, in addition to (17), the infinite series in (15) converge if λ_k, w and W are replaced by $|\lambda_k|$, $|w|$ and $|W|$, respectively. Indeed, this modified series converges if $|w||W|^N/(1-|W|)^N < R$, that is, if $|W| < ((|w|/R)^{1/N}+1)^{-1}$. By (16), this last inequality is satisfied if

(19) $$|t| < r/K\{(|w|/R)^{1/N} + 1\}.$$

Thus, if (17) and (19) hold, by interchanging the order of summation in (18) we obtain

(20) $$\Delta = \frac{1}{\zeta - z} \sum_{n=0}^{\infty} A_n W^n,$$

where A_n and W are given by (4) and (16), respectively.

Since the series on the right-hand side of (20) is uniformly convergent for ζ on C, therefore on substituting it into (14) we may interchange the order of summation and integration to get

$$\Omega = \sum_{n=0}^{\infty} A_n \frac{1}{2\pi i} \int_C F(\zeta) W^n \frac{d\zeta}{\zeta - z} = \sum_{n=0}^{\infty} A_n f_n(z) t^n,$$

by virtue of (16) and (10).

This last equation, together with (11), gives us (5), provided $|t|$ is constrained by (9), (13), (17), and (19), that is, t lies in some neighbourhood of the origin. Evidently formula (5) may then be extended by analytic continuation on t as far as possible.

2. **Applications to classical orthogonal polynomials.** In this section we consider several familiar instances of special functions each of which satisfies a Rodrigues formula of type (1). Indeed, in the notations used by Szegö [4] and others (cf., e.g., [2, Chapter X]), we have

(21) $$f_n(z) = (n!)^{-1}(-1)^n \exp(-z^2) H_n(z) = (n!)^{-1} D_z^n\{\exp(-z^2)\},$$

(22) $$f_n(z) = z^\alpha e^{-z} L_n^{(\alpha)}(z) = (n!)^{-1} D_z^n\{z^{n+\alpha} e^{-z}\},$$

(23) $$f_n(z) = z^{\alpha - n} e^{-z} L_n^{(\alpha - n)}(z) = (n!)^{-1} D_z^n\{z^\alpha e^{-z}\},$$

(24) $$f_n(z) = 2^n (z-1)^\alpha (z+1)^{\beta - n} P_n^{(\alpha, \beta - n)}(z) = (n!)^{-1} D_z^n\{(z-1)^{n+\alpha}(z+1)^\beta\},$$

(25) $$\begin{aligned} f_n(z) &= 2^n (z-1)^{\alpha - n} (z+1)^\beta P_n^{(\alpha - n, \beta)}(z) \\ &= (n!)^{-1} D_z^n\{(z+1)^{n+\beta}(z-1)^\alpha\}, \end{aligned}$$

(26) $$\begin{aligned} f_n(z) &= 2^n (z-1)^{\alpha - n} (z+1)^{\beta - n} P_n^{(\alpha - n, \beta - n)}(z) \\ &= (n!)^{-1} D_z^n\{(z-1)^\alpha (z+1)^\beta\}, \end{aligned}$$

for the classical orthogonal polynomials of Hermite, Laguerre, and Jacobi.

On the other hand, for the Bessel polynomials $y_n(z, \alpha - n, \beta)$ it is known that [3, p. 111, Equation (47)]

$$(27)\quad f_n(z) = \frac{\beta^n z^{\alpha - n - 2} e^{-\beta/z}}{n!} y_n(z, \alpha - n, \beta) = \frac{1}{n!} D_z^n \{z^{n+\alpha-2} e^{-\beta/z}\}.$$

The result of the preceding section would apply also to the ultraspherical polynomials $P_n^{\alpha}(z)$, the Legendre polynomials $P_n(z)$, and the Tchebycheff polynomials $U_n(z)$ of the second kind, since

$$(28)\quad f_n(z) = (-1)^n (z^2 - 1)^{-\alpha - n/2} P_n^{\alpha}\left(\frac{z}{\sqrt{(z^2 - 1)}}\right) = \frac{1}{n!} D_z^n \{(z^2 - 1)^{-\alpha}\}$$

and

$$(29)\quad P_n(z) = P_n^{1/2}(z), \qquad U_n(z) = P_n^1(z).$$

Applying (5) to the special functions involved in equations (21) through (28) above, we shall be led to the following generating-function equivalences.

$$(30)\quad \sum_{n=0}^{\infty} A_n H_n(z) \frac{t^n}{n!} = \exp(2zt - t^2) \sum_{k=0}^{\infty} \lambda_k H_{Nk}(z - t) \frac{(wt^N)^k}{(Nk)!}.$$

$$(31)\quad \begin{aligned} \sum_{n=0}^{\infty} A_n L_n^{(\alpha)}(z) t^n &= (1 - t)^{-1-\alpha} e^{-zt/(1-t)} \\ &\cdot \sum_{k=0}^{\infty} \lambda_k L_{Nk}^{(\alpha)}\left(\frac{z}{1 - t}\right)\left(\frac{wt^N}{(1 - t)^N}\right)^k. \end{aligned}$$

$$(32)\quad \sum_{n=0}^{\infty} A_n L_n^{(\alpha - n)}(z) t^n = (1 + t)^{\alpha} e^{-zt} \sum_{k=0}^{\infty} \lambda_k L_{Nk}^{(\alpha - Nk)}(z(1 + t))\left(\frac{wt^N}{(1 + t)^N}\right)^k.$$

$$(33)\quad \begin{aligned} \sum_{n=0}^{\infty} A_n P_n^{(\alpha, \beta - n)}(z) t^n &= (1 - t)^{\beta} \{1 - \tfrac{1}{2}(z + 1)t\}^{-1-\alpha-\beta} \\ &\cdot \sum_{k=0}^{\infty} \lambda_k P_{Nk}^{(\alpha, \beta - Nk)}\left(\frac{z - (z + 1)t/2}{1 - (z + 1)t/2}\right)\left(\frac{wt^N}{(1 - t)^N}\right)^k. \end{aligned}$$

$$(34)\quad \begin{aligned} \sum_{n=0}^{\infty} A_n P_n^{(\alpha - n, \beta)}(z) t^n &= (1 + t)^{\alpha} \{1 - \tfrac{1}{2}(z - 1)t\}^{-1-\alpha-\beta} \\ &\cdot \sum_{k=0}^{\infty} \lambda_k P_{Nk}^{(\alpha - Nk, \beta)}\left(\frac{z + (z - 1)t/2}{1 - (z - 1)t/2}\right)\left(\frac{wt^N}{(1 + t)^N}\right)^k. \end{aligned}$$

$$\sum_{n=0}^{\infty} A_n P_n^{(\alpha-n,\beta-n)}(z)t^n = \{1 + \tfrac{1}{2}(z+1)t\}^{\alpha}\{1 + \tfrac{1}{2}(z-1)t\}^{\beta}$$

$$(35) \qquad \cdot \sum_{k=0}^{\infty} \lambda_k P_{Nk}^{(\alpha-Nk,\beta-Nk)}(z + \tfrac{1}{2}(z^2-1)t)\left(\frac{wt^N}{\{1+(z+1)t/2\}^N\{1+(z-1)t/2\}^N}\right)^k.$$

$$\sum_{n=0}^{\infty} A_n y_n(z, \alpha - n, \beta)\frac{t^n}{n!} = (1 - zt/\beta)^{1-\alpha} e^t$$

$$(36) \qquad \cdot \sum_{k=0}^{\infty} \lambda_k y_{Nk}\left(\frac{z}{1 - zt/\beta}, \alpha - Nk, \beta\right)\frac{(wt^N)^k}{(Nk)!}.$$

$$(37) \qquad \sum_{n=0}^{\infty} A_n P_n^{\alpha}(z)t^n = \mu^{-2\alpha} \sum_{k=0}^{\infty} \lambda_k P_{Nk}^{\alpha}\left(\frac{z-t}{\mu}\right)(w(t/\mu)^N)^k,$$

where, for convenience,

$$(38) \qquad \mu = \sqrt{(1 - 2zt + t^2)}.$$

3. **Concluding remarks.**

Remark 1. Since [4, p. 103]

$$(39) \qquad L_n^{(\alpha)}(z) = \lim_{\beta\to\infty} P_n^{(\alpha,\beta)}(1 - 2z/\beta), \qquad n = 0, 1, 2, \ldots,$$

the generating-function equivalence (32) may formally be derived as a limiting case of (34).

Remark 2. In view of the known relationship (cf., e.g., [2, p. 226, Equation (16)])

$$(40) \qquad L_n^{(\alpha-n)}(z) = (-z)^n (n!)^{-1} c_n(\alpha; z),$$

equation (32) can easily be restated in terms of the Charlier polynomials $c_n(x; \alpha)$ defined by [op. cit., Equation (4)]

$$(41) \quad c_n(x; \alpha) = \sum_{k=0}^{n} (-1)^k \binom{n}{k}\binom{x}{k} k!\alpha^{-k}, \qquad \alpha > 0, \quad x = 0, 1, 2, \ldots .$$

Remark 3. Results (33), (34) and (35) are essentially equivalent. Indeed, it is well known that [2, p. 169]

$$(42) \qquad P_n^{(\alpha,\beta)}(z) = (-1)^n P_n^{(\beta,\alpha)}(-z), \qquad n = 0, 1, 2, \ldots,$$

and that (cf. [4, p. 64])

$$P_n^{(\alpha,\beta-n)}(z) = \left(\frac{1-z}{2}\right)^n P_n^{(-\alpha-\beta-n-1,\beta-n)}\left(\frac{z+3}{z-1}\right). \tag{43}$$

Remark 4. Since it is easily verified that

$$y_n(z, \alpha - n, \beta) = n!(-z/\beta)^n L_n^{(1-\alpha-n)}(\beta/z), \tag{44}$$

it is not difficult to deduce result (36) from (32), and *vice versa.*

Remark 5. By virtue of (29), the generating-function equivalence (37), with $\alpha = \frac{1}{2}$ and $\alpha = 1$, will evidently yield the corresponding results for the Legendre polynomials and the Tchebycheff polynomials of the second kind.

Remark 6. An equivalence theorem for the Meixner polynomials $m_n(x; \beta, c)$ would follow fairly readily from (33) by appealing to the known result [2, p. 226]

$$m_n(x; \beta, c) = n! P_n^{(\beta-1,-\beta-n-x)}(2/c - 1). \tag{45}$$

The details may be omitted.

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REFERENCES

1. F. Brafman, *Generating functions and associated Legendre polynomials*, Quart. J. Math. Oxford Ser. (2) 10 (1959), 156–160. MR 21 #6450.

2. A. Erdélyi, W. Magnus, F. Oberhettinger and F. G. Tricomi, *Higher transcendental functions*. Vol. II, McGraw-Hill, New York, 1953. MR 15, 419.

3. H. L. Krall and O. Frink, *A new class of orthogonal polynomials: The Bessel polynomials*, Trans. Amer. Math. Soc. 65 (1949), 100–115. MR 10, 453.

4. G. Szegö, *Orthogonal polynomials*, Amer. Math. Soc. Colloq. Publ., vol. 23, Amer. Math. Soc., Providence, R. I., 1939. MR 1, 14.

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