ON THE DISTANCE BETWEEN ZEROES
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ABSTRACT. For the equation \( x'' + q(t)x = 0 \), let \( x(t) \) be a solution with consecutive zeroes at \( t = a \) and \( t = b \). A simple inequality is proven that relates not only \( a \) and \( b \) to the integral of \( q^+(t) \) but also any point \( c \in (a, b) \) where \( |x(t)| \) is maximized. As a corollary, it is shown that if the above equation is oscillatory and if \( q(t) \in L^p[0, \infty), 1 < p < \infty \), then the distance between consecutive zeroes must become unbounded.

Consider the following second order linear differential equation:

\[
(1) \quad x''(t) + q(t)x(t) = 0, 
\]

where \( q(t) \) is continuous on some appropriate \( t \) interval. Let \( q^+(t) = \max[q(t), 0] \). Pertaining to (1), the following theorem of Hartman [3, p. 345] is known.

**Theorem 1.** Let \( q(t) \) be real-valued and continuous for \( a < t < b \). If \( x(t) \) is a solution of (1) with two zeroes in \([a, b]\), then

\[
(2) \quad \int_a^b (t - a)(b - t)q^+(t) \, dt > (b - a).
\]

Since \( (b - a)^2/4 \geq (t - a)(b - t) \) for \( t \in (a, b) \), equation (2) \( \Rightarrow \) that

\[
(3) \quad \frac{(b - a)^2}{4} \int_a^b q^+(t) \, dt > (b - a),
\]

or

\[
(4) \quad \int_a^b q^+(t) \, dt > \frac{4}{b - a}.
\]

Thus Theorem 1 has as a corollary the following condition of Lyapunov. Again, see Hartman [3, p. 345].

**Corollary 1.** A necessary condition for any solution \( x(t) \) of (1) to have two zeroes in \([a, b]\) is that \( \int_a^b q^+(t) \, dt > 4/(b - a) \).

The lemma that we would like to present is the following.

**Lemma 1.** Let \( x(t) \) be a solution of (1), where \( x(a) = x(b) = 0 \), and

\[
\int_a^b q^+(t) \, dt > \frac{4}{b - a}.
\]
Let $c$ be a point in $(a, b)$ where $|x(t)|$ is maximized.

Then

(i) $\int_a^c q^{-}(t) \, dt > 1/(c - a),$

(ii) $\int_c^b q^{+}(t) \, dt > 1/(b - c),$

(iii) $\int_a^b q^{+}(t) \, dt > (b - a)/[(b - c)(c - a)].$

Proof. Integrating (1) yields

$$x'(t) - x'(c) = \int_c^t q^{-}(s)x(s) \, ds - \int_c^t q^{+}(s)x(s) \, ds.$$  

Note that $x'(c) = 0$. Another integration gives

$$x(t) - x(c) = \int_c^t (t - s)q^{-}(s)x(s) \, ds - \int_c^t (t - s)q^{+}(s)x(s) \, ds.$$  

Let $t = b$, so that $x(b) = 0$. Equation (5) implies that

$$x(b) - x(c) = \int_c^b (b - s)q^{-}(s)x(s) \, ds - \int_c^b (b - s)q^{+}(s)x(s) \, ds,$$

or

$$x(c) + \int_c^b (b - s)q^{-}(s)x(s) \, ds = \int_c^b (b - s)q^{+}(s)x(s) \, ds.$$

W.L.O.G., we may assume $x(t) \geq 0$, $t \in [a, b]$. Thus we have

$$x(c) \leq \int_c^b (b - s)q^{+}(s)x(s) \, ds < (b - c)\int_c^b q^{+}(s)x(s) \, ds$$

$$\Rightarrow 1 < (b - c)\int_c^b q^{+}(s) \, ds, \quad \text{since } x(s) \leq x(c), \text{ if } s \in [a, b],$$

$$\Rightarrow \int_c^b q^{+}(t) \, dt > \frac{1}{b - c}.$$  

This proves part (ii). Part (i) follows in a similar fashion, except that in equation (5), one now replaces $t$ by $a$. The sum of (i) and (ii) yields part (iii), which completes the lemma.

One way to view Lemma 1 is that it imposes some restrictions on the location of the point $c$ and thus the maximum of $|x(t)|$ in $[a, b]$. That is, $\int_a^b q^{+}(t) \, dt$ is a finite number. But

$$\lim_{c \to a^+} \frac{b - a}{(b - c)(c - a)} = \lim_{c \to b^-} \frac{b - a}{(b - c)(c - a)} = \infty.$$  

Thus $c$ cannot be "too close" to $a$ or $b$. Also, it is interesting to note that $(b - a)/[(b - c)(c - a)] \geq 4/(b - a)$. This means that under the hypothe-
ses of Lemma 1, Corollary 1 follows from Lemma 1.

As a consequence of Lemma 1 (also Theorem 1 or Corollary 1), we have

**Theorem 2.** Suppose \( q^+(t) \in L^p[0, \infty) \), \( 1 \leq p < \infty \). If (1) is oscillatory and if \( x(t) \) is any solution, then the distance between consecutive zeroes of \( x(t) \) must become infinite.

**Proof.** Suppose not. Then there exists a solution \( x(t) \) with its sequence of zeroes \( \{t_n\} \), which sequence has a subsequence \( \{t_{n_k}\} \) such that

\[ |t_{n_k+1} - t_{n_k}| \leq M < \infty \quad \forall k. \]

Let \( s_{n_k} \) be a point in \((t_{n_k}, t_{n_k+1})\) where \(|x(t)|\) is maximized. Then \(|s_{n_k} - t_{n_k}| < M\), for all \( k \). Since \( q^+(t) \in L^p[0, \infty) \), \( 1 \leq p < \infty \), choose \( k \) so large that

\[
\left( \int_{t_{n_k}}^{\infty} q^+(t)^p \, dt \right)^{1/p} \leq M^{-1-1/r}, \quad \text{where} \quad \frac{1}{p} + \frac{1}{r} = 1.
\]

From Lemma 1, part (i), we have

\[
\int_{t_{n_k}}^{s_{n_k}} q^+(t) \, dt > \frac{1}{s_{n_k} - t_{n_k}}.
\]

Thus

\[
1 < (s_{n_k} - t_{n_k}) \int_{t_{n_k}}^{s_{n_k}} q^+(t) \, dt
\]

\[
< (s_{n_k} - t_{n_k}) \left( \int_{t_{n_k}}^{s_{n_k}} q^+(t)^p \, dt \right)^{1/p} (s_{n_k} - t_{n_k})^{1/r}
\]

\[
< (s_{n_k} - t_{n_k})^{1+1/r} \left( \int_{t_{n_k}}^{\infty} q^+(t)^p \, dt \right)^{1/p}
\]

\[
< M^{1+1/r} \cdot M^{-1-1/r} = 1 < 1,
\]

a contradiction. This completes the theorem.

Pertaining to (1), there is the following oscillation theorem of Wintner [5].

If \( \lim_{t \to \infty} \int_0^t q(s) \, ds = \infty \), then (1) is oscillatory.

The above condition enables us to construct some simple examples.

Consider the equation

\[
x'' + (1+t)^{-1} x = 0, \quad t \geq 0.
\]

The Wintner condition guarantees that (6) is oscillatory. Since \( 1/(1+t) \in L^2[0, \infty) \), Theorem 2 asserts that the distance between zeroes of any solution must become unbounded.

As another example, let
\[ q(t) = 1/(n+1), \quad n + 1/n^2 \leq t \leq (n+1) - 1/(n+1)^2, \quad n \geq 2; \]
\[ = (n)^{1/4}, \quad t = n, \quad n \geq 2; \]
\[ = \text{the line segment joining } (n - 1/n^2, 1/n) \text{ to } (n, n^{1/4}) \]
\[ \text{for } n - 1/n^2 \leq t \leq n, \quad n \geq 2; \]
\[ = \text{the line segment joining } (n, n^{1/4}) \text{ to } (n + 1/n^2, 1/(n+1)) \]
\[ \text{for } n \leq t \leq n + 1/n^2, \quad n \geq 2; \]
\[ = 1/2 \quad \text{for } 0 \leq t \leq 7/4. \]

So \( q(t) \) has the following appearance.

\[ \begin{array}{cccccc}
1 & 2 & 3 & 4 & \ldots \\
\end{array} \]

It is easy to verify that \( \int_0^\infty q(t) \, dt = \infty \), but \( \int_0^\infty q(t)^2 \, dt < \infty \). The Wintner condition again implies that (1) is oscillatory, while Theorem 2 implies that the distance between zeroes is unbounded.

Theorem 2 can also be used to derive a known limit point result (Patula and Wong [4, p. 10, Corollary]). Note that for \( t \geq 0 \), equation (1) is called limit point, L.P., if at least one solution \( x(t) \not\in L^2[0, \infty) \). If any two linearly independent (and thus all) solutions are square integrable, (1) is called limit circle, L.C. See Coddington and Levinson [1, p. 225].

The following lemma is known (Patula and Wong [4, p. 11]).

Lemma 2. If equation (1) is L.C., then (1) is oscillatory, and the distance between consecutive zeroes of any solution tends to zero, as \( t \to \infty \).

We can now prove the following limit point result.

Corollary 2. If \( q^+(t) \in L^p[0, \infty), 1 \leq p < \infty \), then (1) is in the limit point classification.

Proof. Suppose not. Then (1) is L.C. Let \( x(t) \) be any solution of (1). By Lemma 2, \( x(t) \) oscillates and the distance between consecutive zeroes of any solution tends to zero, as \( t \to \infty \). However, Theorem 2 maintains that if \( x(t) \) oscillates, the distance between consecutive zeroes must become unbounded, a contradiction. Thus the equation must be limit point.

It should be noted that Theorem 2 does not hold for \( p = \infty \), as evidenced by the simple example \( x'' + x = 0 \). However, it would be interesting to know
if Theorem 2 is true for $0 < p < 1$. If it were, then Corollary 2 could also be extended to the case $0 < p < 1$. This would answer a question posed by Everitt, Giertz, and Weidmann [2, p. 346] as to whether or not (1) is limit point for $q^+(t) \in L^p[0, \infty)$, $0 < p < 1$.


REFERENCES


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