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OSCILLATION PROPERTIES OF PERTURBED DISCONJUGATE EQUATIONS

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ABSTRACT. Oscillation conditions are given for the equation $Lu + f(t, u) = 0$, where

$$Lu = \frac{1}{\beta_n}\frac{d}{dt}\frac{1}{\beta_{n-1}}\cdots\frac{d}{dt}\frac{1}{\beta_1}\frac{d}{dt}\frac{u}{\beta_0} \qquad (n \geq 2),$$

with $\beta_0, \ldots, \beta_n$ positive and continuous on $(0, \infty)$, $\int^\infty \beta_i\,dt = \infty$ $(1 \leq i \leq n-1)$, and f subject to conditions which include $uf(t, u) \geq 0$. The results obtained include previously known oscillation conditions for the equation $u^{(n)} + f(t, u) = 0$ for both linear and nonlinear cases.

We study the oscillation properties of the scalar equation

$$Lu + f(t, u) = 0, \qquad t > 0, \tag{1}$$

where

$$L = \frac{1}{\beta_n}\frac{d}{dt}\frac{1}{\beta_{n-1}}\cdots\frac{d}{dt}\frac{1}{\beta_1}\frac{d}{dt}\frac{\cdot}{\beta_0} \qquad (n \geq 2), \tag{2}$$

with $\beta_0, \ldots, \beta_n$ continuous and positive on $(0, \infty)$. If $p_1, \ldots, p_n$ are continuous on $(0, \infty)$ and the equation

$$Lx \equiv x^{(n)} + p_1(t)x^{(n-1)} + \cdots + p_n(t)x = 0 \tag{3}$$

is disconjugate on $(0, \infty)$, then L can be rewritten as in (2), with $\beta_i \in C^{(n-i)}(0, \infty)$ [11]. Thus, the class of operators (2) properly contains the *normal disconjugate operators* (3), as defined in [4].

Nehari [10] and Bogar [2] have studied the oscillation properties of (1) for the linear case, under assumptions that certain iterated integrals involving $\beta_1, \ldots, \beta_{n-1}$ diverge as $t \to \infty$. These assumptions are implied by the condition

$$\int^\infty \beta_i\,dt = \infty, \qquad 1 \leq i \leq n-1, \tag{4}$$

which was assumed by Kartsatos [5] in connection with a nonlinear case, and which, the author showed in [13], may be imposed without loss of generality; i.e., (2) can always be rewritten so that (4) holds, and this requirement determines $\beta_0, \ldots, \beta_n$ uniquely up to positive multiplicative constants

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with product one. Henceforth we assume (4).

If $\alpha_1, \alpha_2, \ldots$ are continuous on $(0, \infty)$ and $c > 0$, define $I_0 = 1$ and

$$I_r(t, c; \alpha_r, \ldots, \alpha_1) = \int_c^t \alpha_r(\lambda) I_{r-1}(\lambda, c; \alpha_{r-1}, \ldots, \alpha_1)\, d\lambda, \qquad r \geq 1. \tag{5}$$

Repeated integration by parts yields

$$I_r(t, c; \alpha_r, \ldots, \alpha_1) = (-1)^r I_r(c, t; \alpha_1, \ldots, \alpha_r) \tag{6}$$

[14, Lemma 2.2], which we will use later.

The functions

$$\tilde{x}_j(t) = \beta_0(t) I_{j-1}(t, c; \beta_1, \ldots, \beta_{j-1}), \qquad 1 \leq j \leq n,$$

form a fundamental set for $Lx = 0$ such that $\tilde{x}_j(t) > 0$ for large t, and, because of (4), $\lim_{t\to\infty} \tilde{x}_i(t)/\tilde{x}_j(t) = 0$, $1 \leq i < j \leq n$. Such a fundamental set is called a *principal system* for $Lx = 0$. It has been shown by Hartman [4], Levin [7] and Willett [14] that a normal disconjugate equation (3) has principal systems, and the author showed [13] (without assuming (4) at the outset) that this is also true of $Lx = 0$ with L as in (2).

The adjoint of (2) is

$$L^* = \frac{1}{\beta_0} \frac{d}{dt} \frac{1}{\beta_1} \cdots \frac{d}{dt} \frac{1}{\beta_{n-1}} \frac{d}{dt} \frac{\cdot}{\beta_n}$$

(Nehari [10]), and the functions

$$\tilde{y}_j(t) = \beta_n(t) I_{j-1}(t, c; \beta_{n-1}, \ldots, \beta_{n-j+1}), \qquad 1 \leq j \leq n,$$

form a principal system for $L^* y = 0$.

We define the operators $L_0, \ldots, L_n$ by $L_0 x = x/\beta_0$ and

$$L_j x = \frac{1}{\beta_j} \frac{d}{dt} \frac{1}{\beta_{j-1}} \cdots \frac{d}{dt} \frac{1}{\beta_1} \frac{d}{dt} \frac{x}{\beta_0}, \qquad 1 \leq j \leq n,$$

so that

$$(L_{j-1} x)' = \beta_j L_j x, \qquad 1 \leq j \leq n, \tag{7}$$

and $L_n = L$. An *extendible* solution of (1) is a function u such that $L_0 u, \ldots, L_n u$ exist and satisfy (1) on some interval (t_0, ∞). An extendible solution is *oscillatory* if it has infinitely many zeros on every such interval.

Lemma 1. *Suppose* $u(t) > 0$ *and* $L_n u(t) \leq 0$ *for* $t \geq t_0$. *Then*

(a) *if* n *is even, there is an even integer* k *such that* $0 \leq k \leq n - 2$,

$$L_j u(t) > 0 \quad \textit{for large } t, \qquad 0 \leq j \leq k, \tag{8}$$

$$\lim_{t\to\infty} L_{k+1} u(t) = \alpha \qquad (\textit{finite}, \geq 0), \tag{9}$$

and

(10) $$\lim_{t\to\infty} L_j u(t) = 0, \qquad k+2 \le j \le n-1;$$

(b) *if* n *is odd, either*

(11) $$\lim_{t\to\infty} L_j u(t) = 0, \qquad 1 \le j \le n-1,$$

or there is an odd integer k *such that* (8), (9) *and* (10) *hold.*

This lemma is an adaptation of a result of Kneser [6]. Its proof is by separate inductions for even and odd n, and leans heavily on the fact that if $(-1)^s \lim_{t\to\infty} L_j u(t) \ge \epsilon > 0$ for some $j \ge 1$, then (4) and repeated integration of (7) yield $(-1)^s \lim_{t\to\infty} L_0 u(t) = \infty$.

Let $\{x_1, \ldots, x_n\}$ and $\{y_1, \ldots, y_n\}$ be principal systems for $Lx = 0$ and $L^* y = 0$, and let z_i be a solution of $L_i(x_1 z_i) = y_{n-i}/y_1$, $0 \le i \le n-2$. It can be shown that if $t_1 > 0$, then

(12) $$x_i(t) = \beta_0(t) \sum_{j=1}^{i} a_{ij} I_{j-1}(t, t_1; \beta_1, \ldots, \beta_{j-1})$$

and

(13) $$y_i(t) = \beta_n(t) \sum_{j=1}^{i} b_{ij} I_{j-1}(t, t_1; \beta_{n-1}, \ldots, \beta_{n-j+1}),$$

where a_{ij} and b_{ij} are constants, $a_{ii} > 0$ and $b_{ii} > 0$. From (4), (12), (13) and L'Hospital's rule,

(14) $$\lim_{t\to\infty} \frac{\beta_0(t) I_{i-1}(t, t_1; \beta_1, \ldots, \beta_{i-1})}{x_i(t)} = a_{ii}^{-1} > 0$$

and

(15) $$\lim_{t\to\infty} \frac{\beta_n(t) I_{i-1}(t, t_1; \beta_{n-1}, \ldots, \beta_{n-i+1})}{y_i(t)} = b_{ii}^{-1} > 0;$$

moreover, it can be shown that

(16) $$\lim_{t\to\infty} \frac{I_{n-1}(t, t_1; \beta_{n-1}, \ldots, \beta_1)}{z_0(t)} = c_0 > 0 \qquad \text{(finite)}$$

and

(17) $$\lim_{t\to\infty} \frac{I_{n-1}(t, t_1; \beta_1, \ldots, \beta_k, \beta_{n-1}, \ldots, \beta_{k+1})}{z_k(t)} = c_k > 0 \quad \text{(finite)},$$

$$1 \le k \le n-2.$$

The last four equations imply that if the assumptions of Theorems 1 and 2 hold for a given choice of principal systems for $Lx = 0$ and $L^* y = 0$, then they hold for all choices.

Theorem 1. *Suppose f is continuous on $(0, \infty) \times (-\infty, \infty)$,*

$$uf(t, u) \geq 0, \tag{18}$$

and

$$|f(t, u)| \geq h(t)\phi(|u|/\beta_0(t)), \tag{19}$$

where h is continuous and nonnegative and ϕ is nondecreasing and positive on $(0, \infty)$. Suppose also that

$$\int^{\infty} y_{n-k-1}(s)h(s)\phi\left(\frac{\rho x_{k+1}(s)}{x_1(s)}\right)ds = \infty \tag{20}$$

for every positive ρ and nonnegative integer k such that $n-k$ is even and positive. Then (a) *if n is even, every extendible solution of* (1) *is oscillatory;* (b) *if n is odd and u is a nonoscillatory extendible solution of* (1), *then $\lim_{t\to\infty} L_j u(t) = 0$ $(1 \leq j \leq n-1)$ and $\lim_{t\to\infty} L_0 u(t) = \gamma$ (finite). In this case, $\gamma = 0$ if*

$$\int^{\infty} y_n(s)h(s)\,ds = \infty. \tag{21}$$

Proof. Suppose u is an extendible solution of (1) which is positive on (t_0, ∞) and satisfies (8), (9) and (10) for the integer k indicated in Lemma 1. Then (7), (9) and (10) yield

$$L_{k+1}u(t) = \alpha + \int_t^{\infty} I_{n-k-2}(t, s; \beta_{k+2}, \ldots, \beta_{n-1})\beta_n(s)f(s, u(s))\,ds, \qquad t \geq t_0, \tag{22}$$

and, since $n-k-2$ is even, (18) implies that the integrand is nonnegative; therefore,

$$L_{k+1}u(t) \geq \alpha, \qquad t \geq t_0. \tag{23}$$

The absolute convergence of the integral in (22), together with (6), (15) and (19), implies that

$$\int^{\infty} y_{n-k-1}(s)h(s)\phi(L_0 u(s))\,ds < \infty. \tag{24}$$

But (7) and (23) imply that $(L_k u)' \geq 0$; hence, from (8), $L_k u(t) \geq R$ for some positive R and t sufficiently large, say $t \geq t_1$. Repeated integration of (7) for $j = k, \ldots, 1$ yields

$$L_0 u(t) \geq RI_k(t, t_1; \beta_1, \ldots, \beta_k) + \sum_{j=0}^{k-1} L_j u(t_1) I_j(t, t_1; \beta_1, \ldots, \beta_j), \qquad t \geq t_1. \tag{25}$$

From (4) and L'Hospital's rule,

$$\lim_{t\to\infty} \frac{I_j(t, t_1; \beta_1, \ldots, \beta_j)}{I_k(t, t_1; \beta_1, \ldots, \beta_k)} = 0, \qquad 0 \le j < k;$$

hence, (25) implies that

$$L_0 u(t) \ge (R/2) I_k(t, t_1; \beta_1, \ldots, \beta_k) \quad \text{for large } t,$$

and therefore, from (14),

$$\liminf_{t\to\infty} \frac{L_0 u(t)}{x_{k+1}(t)/x_1(t)} > 0.$$

This and (24) imply that the integral in (20) converges for some positive ρ, a contradiction.

It now follows from Lemma 1 that (1) has no extendible positive solutions except, possibly, some which satisfy (11), if n is odd. For such a solution, $(-1)^{n-j} L_j u(t) \le 0$ $(1 \le j \le n)$ for large t, so that $L_1 u(t) \le 0$ and $L_0 u(t)$ is decreasing and positive for large t; hence,

$$L_0 u(t) \ge \gamma = \lim_{t\to\infty} L_0 u(t) \ge 0. \tag{26}$$

From (7), (11) and (26),

$$L_0 u(t) = \gamma + \int_t^\infty I_{n-1}(t, s; \beta_1, \ldots, \beta_{n-1}) \beta_n(s) f(s, u(s))\, ds. \tag{27}$$

If $\gamma > 0$, then $f(s, u(s)) \ge h(s)\phi(\gamma)$, and the convergence of the integral in (27) implies that

$$\int_t^\infty I_{n-1}(t, s; \beta_1, \ldots, \beta_{n-1}) \beta_n(s) h(s)\, ds < \infty,$$

which, because of (6) and (15), implies that the integral in (21) converges.

This completes the proof of Theorem 1 as far as positive solutions are concerned. Since f satisfies (18) and Lemma 1 also holds with all of its inequalities reversed, the discussion of negative solutions proceeds in the same way.

Theorem 1 contains the following known special case.

Corollary 1 (Anan'eva and Balaganskiĭ [1]). *If p is positive and continuous on $(0, \infty)$ and*

$$\int^\infty s^{n-2} p(s)\, ds = \infty, \tag{28}$$

then every solution of $y^{(n)} + p(t)y = 0$, $t > 0$, is oscillatory if n is even and, if n is odd, every nonoscillatory solution must approach zero monotonically as $t \to \infty$, along with its first $n - 1$ derivatives.

(Anan'eva and Balaganskiĭ also considered in [1] the oscillation properties of (1), subject to (2), (4) and (18), but with other conditions on f different from ours, and obtained a result which contains Fite's oscillation theorem [3, Theorem V], but not Corollary 1.)

The next theorem is related to results of Ryder and Wend [12] for the special case where $Lu = u^{(n)}$, under assumptions which rule out linearity for (1).

Theorem 2. *Suppose f satisfies the assumptions of Theorem 1, except for (20). Then the conclusions of Theorem 1 hold if* (a)

$$\int_{\rho}^{\infty} (\phi(r))^{-1}\, dr < \infty, \qquad 0 < \rho < \infty, \tag{29}$$

and

$$\int^{\infty} y_1(s) z_k(s) h(s)\, ds = \infty \tag{30}$$

for all integers k such that $n - k$ is even and positive; *or if* (b)

$$\phi(r_1 r_2) \geq \phi(r_1)\phi(r_2), \qquad r_1, r_2 > 0, \tag{31}$$

$$\int_0^{\rho} (\phi(r))^{-1}\, dr < \infty, \qquad 0 < \rho < \infty, \tag{32}$$

and

$$\int^{\infty} y_1(s) h(s) \phi(z_k(s))\, ds = \infty \tag{33}$$

for all such k.

Proof. As in the proof of Theorem 1, it suffices to consider positive solutions. Suppose u is a positive solution of (1) on (t_0, ∞) which satisfies (8), (9) and (10) for the integer k indicated in Lemma 1. Since $\alpha \geq 0$, (19) and (22) imply that

$$L_{k+1} u(t) \geq \int_t^{\infty} I_{n-k-2}(t, s; \beta_{k+2}, \ldots, \beta_{n-1}) V(s)\, ds, \qquad t \geq t_0, \tag{34}$$

where

$$V(s) = \beta_n(s) h(s) \phi(L_0 u(s)). \tag{35}$$

Now suppose (8) is satisfied for $t \geq t_1$. Multiplying (34) by β_{k+1}, integrating from t_1 to t, and changing the order of integration yields

$$\begin{aligned} L_k u(t) > &\int_{t_1}^{t} V(s)\, ds \int_{t_1}^{s} \beta_{k+1}(\lambda) I_{n-k-2}(\lambda, s; \beta_{k+2}, \ldots, \beta_{n-1})\, d\lambda \\ &+ \int_t^{\infty} V(s)\, ds \int_{t_1}^{t} \beta_{k+1}(\lambda) I_{n-k-2}(\lambda, s; \beta_{k+2}, \ldots, \beta_{n-1})\, d\lambda, \qquad t \geq t_1. \end{aligned} \tag{36}$$

Since $n - k - 2$ is even,

$$I_{n-k-2}(\lambda, s; \beta_{k+2}, \ldots, \beta_{n-1}) \geq I_{n-k-2}(\lambda, t; \beta_{k+2}, \ldots, \beta_{n-1}), \quad \lambda \leq t \leq s.$$

Substituting the right side in the second integral in (36) and using (5) and (6) yields

$$\text{(37)} \quad \begin{aligned} L_k u(t) &> \int_{t_1}^{t} I_{n-k-1}(s, t_1; \beta_{n-1}, \ldots, \beta_{k+1}) V(s)\, ds \\ &\quad + I_{n-k-1}(t, t_1; \beta_{n-1}, \ldots, \beta_{k+1}) \int_t^\infty V(s)\, ds, \qquad t \geq t_1. \end{aligned}$$

If $k = 0$, this gives a lower bound for $L_0 u(t)$; if $k \geq 1$, we ignore the first integral in (37). Then (7), (8) and (37) yield

$$L_{k-1} u(t) > \int_{t_1}^{t} \beta_k(\lambda) I_{n-k-1}(\lambda, t_1; \beta_{n-1}, \ldots, \beta_{k+1})\, d\lambda \int_\lambda^\infty V(s)\, ds.$$

Changing the order of integration and using (5) yields

$$\text{(38)} \quad \begin{aligned} L_{k-1} u(t) &> \int_{t_1}^{t} I_{n-k}(s, t_1; \beta_k, \beta_{n-1}, \ldots, \beta_{k+1}) V(s)\, ds \\ &\quad + I_{n-k}(t, t_1; \beta_k, \beta_{n-1}, \ldots, \beta_{k+1}) \int_t^\infty V(s)\, ds, \qquad t \geq t_1. \end{aligned}$$

If $k = 1$, (38) gives a lower bound for $L_0 u(t)$; if $k > 1$, repetition of the argument that led from (37) to (38) yields finally the inequality

$$\text{(39)} \quad \begin{aligned} L_0 u(t) &> \int_{t_1}^{t} w_k(s) \beta_n(s) h(s) \phi(L_0 u(s))\, ds \\ &\quad + w_k(t) \int_t^\infty \beta_n(s) h(s) \phi(L_0 u(s))\, ds, \qquad t \geq t_1, \end{aligned}$$

after recalling (35) and defining

$$\text{(40)} \quad w_k(t) = \begin{cases} I_{n-1}(t, t_1; \beta_{n-1}, \ldots, \beta_1), & k = 0, \\ I_{n-1}(t, t_1; \beta_1, \ldots, \beta_k, \beta_{n-1}, \ldots, \beta_{k+1}), & 1 \leq k \leq n-2. \end{cases}$$

If (a) of Theorem 2 holds, call the first integral in (39) $F(t)$ and ignore the second. We employ a device used by Macki and Wong [9] and Ryder and Wend [12]: since ϕ is nondecreasing and $F(t) > 0$ for large t (say $t \geq t_2$), (39) implies that $\phi(L_0 u(t))/\phi(F(t)) \geq 1$, $t \geq t_2$. (Since $h(t) \geq 0$ by assumption, $F(t) \equiv 0$ would imply that $h(t) \equiv 0$, contradicting (30).) Therefore,

$$F'(t)/\phi(F(t)) \geq \beta_n(t) w_k(t) h(t), \qquad t \geq t_2,$$

and

$$\int_{F(t_2)}^{F(t)} (\phi(r))^{-1}\, dr \geq \int_{t_2}^{t} \beta_n(s) w_k(s) h(s)\, ds, \qquad t \geq t_2;$$

Hence, (29) implies that the integral on the right converges as $t \to \infty$. However, from (13), (16), (17) and (40),

(41) $$\lim_{s \to \infty} w_k(s)/z_k(s) = c_k \quad \text{and} \quad y_1(s) = b_{11}\beta_n(s),$$

and therefore the integral (30) is convergent, a contradiction. Hence, the conclusions of Theorem 1 hold under the assumptions of Theorem 2(a).

If (b) of Theorem 2 holds, call the second integral in (39) $H(t)$ and ignore the first; then, from (31),

$$\phi(L_0u(t))/\phi(H(t)) \geq \phi(w_k(t)), \qquad t \geq t_1,$$

and therefore

$$-H'(t)/\phi(H(t)) \geq \beta_n(t)h(t)\phi(w_k(t))$$

and

$$\int_{H(t)}^{H(t_1)} (\phi(r))^{-1}\,dr \geq \int_{t_1}^{t} \beta_n(s)h(s)\phi(w_k(s))\,ds.$$

Now (32) implies that the integral on the right converges as $t \to \infty$, which, because of (31) and (41), contradicts (33). Thus, the conclusions of Theorem 1 hold under the assumptions of Theorem 2(b).

For the special case where $Lx = x^{(n)}$, (30) is equivalent to $\int^{\infty} s^{n-1}h(s)\,ds = \infty$, and Theorem 2(a) is related to Theorem 1 of Ryder and Wend [12], and is a generalization of the sufficiency half of Theorem 2 of Ličko and Švec [8] for the equation

(42) $$y^{(n)} + h(t)|y|^{\gamma}\operatorname{sgn} y = 0 \qquad (h > 0),$$

with $\gamma > 1$. The proof of Theorem 2(a) is based on the proof of Ryder and Wend. Because of (31), (33) is equivalent to $\int^{\infty} h(s)\phi(s^{n-1})\,ds = \infty$ if $Lx = x^{(n)}$, so that Theorem 2(b) is related to Theorem 2 of Ryder and Wend (but the proofs are different), and is a generalization of the sufficiency half of Theorem 1 of Ličko and Švec [8] for (42), with $0 < \gamma < 1$.

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