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ON $N^{\aleph_1}$ AND THE ALMOST-LINDELÖF PROPERTY

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ABSTRACT. In 1970, Kemperman and Maharam proved that there exists a Baire measure μ on $N^{\mathbf{c}}$ (where N is the set of natural numbers) such that $N^{\mathbf{c}}$ may be covered by a famliy of elementary open μ-null sets and used this to prove that $R^{\mathbf{c}}$ (where R is the set of real numbers) does not have the "almost-Lindelöf" property. We define K to be the smallest cardinal κ for which there exists a collection of κ closed subsets of R each of Lebesgue measure zero and which covers R, and we show that in the above results $\mathbf{c}$ can be replaced by K. We then note that we have shown elsewhere that it is consistent with the negation of the continuum hypothesis that $\mathrm{K} = \aleph_1$, and this, therefore, implies that it is consistent with the negation of the continuum hypothesis that $R^{\aleph_1}$ not be almost-Lindelöf.

As in [4], a topological space T is defined to be *almost Lindelöf* iff for every Baire measure μ on T and every open cover $\mathcal{G}$ of T there exists a countable family $\mathcal{H} \subseteq \mathcal{G}$ such that $T - \bigcup\mathcal{H}$ is μ-null. In [4] J. H. B. Kemperman and Dorothy Maharam consider the question, due to H. Rubin, as to whether or not $R^{\aleph_1}$ (where R is the real line) is almost Lindelöf. They show that $R^{\mathbf{c}}$ is not almost Lindelöf thus settling the problem when given the continuum hypothesis, and, in fact, they prove a much stronger result. They do this by constructing a certain map from the set I of irrationals between zero and one into the product $N^{\mathbf{c}}$ (where N is the set $\{1, 2, 3, \ldots\}$ of natural numbers), using this map to define a Baire measure and an appropriate cover on $N^{\mathbf{c}}$, and then extending both to $R^{\mathbf{c}}$.

In this paper we shall define an uncountable cardinal $\mathrm{K} \leq \mathbf{c}$, and we shall show how the above construction can be modified to apply to N^{K} and, therefore, to R^{K}. We shall then note that elsewhere [1] we have proven the consistency of $\mathrm{K} = \aleph_1$ with the negation of the continuum hypothesis and thus prove that it is consistent with the negation of the continuum hypothesis that $R^{\aleph_1}$ not be almost Lindelöf.

We begin by letting λ denote Lebesgue measure, and we define a family of closed subsets of R to be a *λ-F-cover* iff it covers R and each of its members has Lebesgue measure zero. Then let K be the smallest cardinal κ for which there exists a λ-F-cover of cardinality κ. That K exists follows from the axiom of choice (which implies that every set of cardinals has a least

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member and which we shall assume without further mention); that it is at most $\mathfrak{c}$, from the fact that the family of all singletons of R forms a λ-F-cover and that it is uncountable, from the fact that each member of the cover has Lebesgue measure zero. Now we modify the construction in [4] to prove

Theorem 1. *There exists a function* ψ *from* I *into* $N^{\mathbf{K}}$ *and an elementary open cover* $\mathcal{G}$ *of* $N^{\mathbf{K}}$ *such that*:

(a) *If* U *is any elementary open subset of* $N^{\mathbf{K}}$, *then* $\psi^{-1}[U]$ *is a Borel set in* I.

(b) *If* G *is any member of* $\mathcal{G}$, *then* $\lambda(\psi^{-1}[G]) = 0$.

Proof. Let $\mathcal{C}^*$ be any λ-F-cover of cardinality $\mathbf{K}$, $\mathcal{C}$ the family $\{C \cap I: C \in \mathcal{C}^* \text{ and } C \cap I \neq \varnothing\}$, and since it is well known that I is isomorphic to the product space N^N (where N has the discrete topology), identify I with this space. Thus a member of I will be an infinite sequence of members of N, and a member of $\mathcal{C}$ will be a set of such sequences. Also, it will be convenient to represent $N^{\mathbf{K}}$ as $N^N \times N^{\mathcal{C}}$ ($R - I$ is countable, so $\mathcal{C}$ must have cardinality $\mathbf{K}$), and so a point in $N^{\mathbf{K}}$ will be a pair (x, y) where $x = (x_1, x_2, \ldots)$ is a sequence of points in N and y is a function from $\mathcal{C}$ into N.

We now define our function $\psi: I \to N^{\mathbf{K}}$ by setting $\psi(x) = (x, y_x)$, where y_x is the function from $\mathcal{C}$ into N defined by

(i) $y_x(C) = 1$ if $x \in C$,

(ii) $y_x(C) = \inf\{n \in N: r \notin C \text{ whenever } r_k = x_k \text{ for } 1 \leq k < n\}$ if $x \notin C$.

We note that in each instance of (ii) the value $y_x(C)$ is well defined and strictly greater than 1 because each $C \in \mathcal{C}$ is closed and nonempty.

To prove that ψ^{-1} applied to elementary open sets yields Borel sets, it is sufficient to note that sets of the form $\{x \in N^N: x_k = n\}$ or $\{x \in N^N: y_x(C) = n > 1\}$ are open, while those of the form $\{x \in N^N: y_x(C) = 1\} = C$ are closed. Thus ψ^{-1} of an elementary open set is, at worst, both $\mathbf{F}_\sigma$ and $\mathbf{G}_\delta$.

Next we define $\mathcal{G}$. For each $r \in C \in \mathcal{C}$ and each $n \in N$ we set

$$G_n(r, C) = \{(x, y) \in N^{\mathbf{K}}: y(C) = n \quad \text{and} \quad k < n \to x_k = r_k\},$$

and we set

$$\mathcal{G} = \{G_n(r, C) : n \in N \quad \text{and} \quad r \in C \in \mathcal{C}\}.$$

To see that $\mathcal{G}$ is a cover, choose any $(x, y) \in N^{\mathbf{K}}$. Then x belongs to I and, therefore, to some $C \in \mathcal{C}$. Thus $(x, y) \in G_{y(C)}(x, C)$.

Finally, we note that

$$\psi^{-1}[G_n(r, C)] = \begin{cases} C & \text{if } n = 1, \\ \varnothing & \text{otherwise,} \end{cases}$$

and that in either case we have $\lambda(\psi^{-1}[G_n(r, C)]) = 0$. □

We note that if $\mathbf{K} = \mathfrak{c}$ and we set $\mathcal{C} = \{\{r\}: r \in I\}$, the above construction reduces to that in [4].

While we do not know if can replace $\mathbf{K}$ by $\aleph_1$ in Theorem 1 when $\mathbf{K}$ is strictly greater than $\aleph_1$, and we do not even have complete information as to all of the possible values of $\mathbf{K}$, it can be seen that the set $\{F_\alpha: \alpha < \kappa\}$ constructed in [1] is a λ-F-cover of cardinality κ, and, therefore, from the related results in [1] that it is consistent with the negation of the continuum hypothesis that $\mathbf{K}$ be any regular uncountable cardinal less than or equal to $\mathfrak{c}$. Thus we have

Theorem 2. *It is consistent with the negation of the continuum hypothesis that $R^{\aleph_1}$ not be almost Lindelöf. In particular, it is consistent that there exists a Baire measure μ on $R^{\aleph_1}$ such that $\mu(R^{\aleph_1}) = 1$ and that there exists an open cover of $R^{\aleph_1}$ by elementary μ-null sets.*

Proof. The proof is exactly as [4] except that we use our function ψ and our cover $\mathcal{G}$ as obtained using Theorem 1. □

We note that a closed set of measure zero must be nowhere dense, so if we let $\mathbf{K}_N$ denote the number of nowhere dense sets needed to cover R, and we let $\mathbf{K}_L$ denote the number of sets of Lebesgue measure zero needed to cover R, then we have $\mathbf{K} \geq \max(\mathbf{K}_N, \mathbf{K}_L)$. It would be interesting to know if this inequality can ever be strict. It is known that given Martin's axiom [5] all three cardinals are equal to $\mathfrak{c}$, and it is also known [6] that if "generic" reals are added, $\mathbf{K}_N$ becomes equal to $\mathfrak{c}$ and $\mathbf{K}_L$ remains equal to $\aleph_1$, while if "random" reals are added, the reverse occurs. We also note that elsewhere [2], [3] we have studied a related cardinal, namely, the cardinality of the smallest compact cover of I.

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