PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 52, October 1975

THE MAPS $BSp(1) \to BSp(n)$

ZAFER MAHMUD

ABSTRACT. Let $Sp(n)$ be the symplectic Lie group. Then it is known that given a map $f: BSp(1) \to BSp(1)$, $f^*: H^4(BSp(1), \mathbf{Z}) \to H^4(BSp(1), \mathbf{Z})$ is zero or multiplication by the square of an odd integer. We generalise the latter part of this result using symplectic K^*-theory.

We begin with some notation. Let $T' \subset Sp(n)$ be the standard maximal torus [2] and $BSp(n)$ a classifying space [8].

All cohomology will have integer coefficients.

From [5], we have $H^*(BT') \cong \mathbf{Z}[t_1, \ldots, t_n]$, dim $t_i = 2$. The inclusion $T' \subset Sp(n)$ induces an injection of $H^*(BSp(n))$ onto the Weyl group invariants in $H^*(BT')$.

Let $T \subset Sp(1)$ be the standard maximal torus. Then we extract the following from [3].

Proposition 1. *If f: $BSp(1) \to BSp(n)$ is a map and f^*: $H^*(BSp(n)) \to H^*(BSp(1))$, then there is an extension ϕ^*: $H^*(BT') \to H^*(BT)$ of f^*.*

Thus, if $H^*(BT) \cong \mathbf{Z}[t]$, then $\phi^* t_i = m(i)t$ for some integer $m(i)$. In this note we prove the following: *In the set $\{m(1), m(2), \ldots, m(n)\}$, each even $m(i)$ occurs an even number of times.* For this purpose we compute $f^!$: $KU^0(BSp(n)) \to KU^0(BSp(1))$, where KU^* is complex K^*-theory.

From [4] we find that $KU^0(BT') \cong \mathbf{Z}[[s_1, \ldots, s_n]]$ where $(1 + s_i)$ is the virtual canonical line bundle over BS^1. Put $z_i = 1 + s_i$. $KU^0(BSp(n))$ is isomorphic to the Weyl group invariants in $KU^0(BT')$ [4], and the Weyl group acts by permuting the z_i and inverting: $z_i \to z_i^{-1}$. Hence $KU^0(BSp)) \cong \mathbf{Z}[[y_1, \ldots, y_n]]$, $y_i = i$th elementary symmetric function in $(z_j + z_j^{-1} - 2)$. For $BSp(1)$, put $y_1 = y$.

Let G be a compact connected Lie group and $R(G)$ its complex representation ring. Then in [4, p. 29], an isomorphism $\hat{\alpha}$: $\hat{R}(G) \to KU^0(BG)$ is described (here $\hat{R}(G)$ is the completion of $R(G)$ under the augmentation topology). There are also monomorphisms α: $R(G) \to KU^0(BG)$ and $R(G) \to \hat{R}(G)$.

If Sp and U are the "big" symplectic and unitary groups, the standard inclusion l: $Sp \to U$ induces a monomorphism l^*: $KSp^*(BSp(n)) \to KU^*(BSp(n))$ of abelian groups. An element of $KU^*(BSp(n))$ is called *symplectic* if it is in the image of l^*.

Received by the editors July 23, 1974.

AMS (MOS) subject classifications (1970). Primary 55F40; Secondary 55F50.

Key words and phrases. Classifying space, Lie group, symplectic K^*-theory, representation.

If $\theta\colon Sp(n) \to Sp(n)$ is the identity representation, then $\alpha(\theta - 2n) = y_1$. Thus y_1 is symplectic and hence so is $f^! y_1$.

Lemma 2. *The image of $l^*\colon KSp^0(BSp(1)) \to KU^0(BSp(1))$ is the subgroup generated by $\{1, y, 2y^2, \dots, y^{2j-1}, 2y^{2j}, \dots\}$.*

Proof. Since y is symplectic, so is y^{2j-1} and since y^{2j} is self-conjugate, $2y^{2j}$ is symplectic. From [7] we deduce that y^{2j} is not symplectic.

So, if $f^! y_1 = \Sigma \gamma(r) y^r$, then $\gamma(2r)$ is even and our 2-primary restrictions on $\{m(j)\}$ arise from this fact.

Proposition 3.

$$\gamma(r) = \sum_{j=1}^{n} \frac{m(j)}{r} \binom{m(j) + r - 1}{2r - 1},$$

where ( ) is the binomial coefficient.

Corollary 4. *For each integer $r \geq 1$,*

$$\sum_{j=1}^{n} \frac{m(j)}{2r} \binom{m(j) + 2r - 1}{4r - 1}$$

is even.

Proof. This is the condition that $\gamma(2r)$ is even.

The proof of Proposition 3 needs

Lemma 5. *Proposition 3 is true for $n = 1$, i.e.*

$$\gamma(r) = \frac{m}{r} \binom{m + r - 1}{2r - 1}, \qquad m(1) = m.$$

Proof. We have the Adams operation $\psi^2\colon KU^0(BSp(1)) \to KU^0(BSp(1))$ [1]. It is natural and $\psi^2 z_i = z_i^2$, $\psi^2 y = 4y + y^2$.

The naturality of ψ^2 gives

$$(*) \qquad \sum \gamma(r)(4y + y^2)^r = 4f^! y + (f^! y)^2.$$

From Proposition 1 and [3] we know a priori that $f^! = \psi^m$. Thus computing $f^!$ amounts to writing $z^m + z^{-m} - 2$ as a polynomial in $(z + z^{-1} - 2)$. This can be done with the help of (*).

Proof of Proposition 3. This follows from Lemma 5 and the equation $\mathrm{Ch}\, f^! y_1 = f^* \mathrm{Ch}\, y_1$, where Ch is the Chern character [4].

Lemma 6. *For any integers m and m' let $m = \Sigma \beta_i 2^i$ and $m' = \Sigma \beta_j' 2^j$ be their 2-adic expansions with $\beta_j, \beta_j' = 0$ or 1. Then*

$$\binom{m}{m'} = \prod_i \binom{\beta_i}{\beta_i'} \mod 2.$$

Proof. Well known.

To get our information on $\{m(i)\}$, we need more notation.

Definition. (i) For any integer $m \neq 0$, write $m = 2^s m'$, where m' is odd and put $\beta(m) = s$.

(ii) Divide $\{m(i)\}$ into disjoint subsets $I_0, I_1, \ldots$, such that $a \in I_s \Rightarrow \beta(a) = s$.

(iii) If in $\{m(j)\}$, $m(i)$ is repeated $d(i)$ times and I_s contains the distinct elements $m(j_1), m(j_2), \ldots$, define $\operatorname{Card} I_s$ to be $d(j_1) + d(j_2) + \ldots$.

(iv) Put

$$C_i(r) = \frac{m(i)}{r}\binom{m(i) + r - 1}{2r - 1}.$$

Lemma 7. (i)

$$C_i(r) = \frac{2m(i)}{m(i) + r}\binom{m(i) + r}{2r}.$$

(ii) *If* $\beta(r) = s$ *and* $m(i) \notin I_{s+1}$, *then* $C_i(2r)$ *is even.*

Proof. (i) Easy from the definition of the binomial coefficient.

(ii)

$$\begin{aligned}\beta(m(i)/(m(i) + 2r)) &= \beta(m(i)) - \beta(m(i) + 2r) \\ &= \beta(m(i)) - (s + 1) \geq 0 \quad \text{if } \beta(m(i)) > s + 1, \\ &= 0 \qquad\qquad\qquad\qquad\quad \text{if } \beta(m(i)) < s + 1.\end{aligned}$$

In either case, $\beta(2m(i)/(m(i) + 2r)) > 0$, and hence $C_i(2r)$ is even.

Proposition 8. (i) *If* I_s *is not empty then* $s > 0 \Rightarrow \operatorname{Card} I_s$ *is even.*

(ii) *If* I_s *is not empty and* $s > 0$, *let the distinct elements of* I_s *for which* $d(\ )$ *is odd be* $m(1), \ldots, m(e^*)$. *Then* [*by part* (i)] $e^* = 2e$ *and there are integers* w_i *and* b_i *with* $b_i = 0$ *or* 1 *such that*

$$m(2i - 1) = 2^s(1 + 4w_i + 2b_{2i-1}),$$

$$m(2i) = 2^s(1 + 4w_i + 2b_{2i}), \qquad i = 1, \ldots, e.$$

Proof. (i) By renumbering if necessary, we can assume that the distinct integers in I_s are the first e' from $\{m(i)\}$. Write $m(i)$ as $m(i) = \sum_{u \geq 0} a(iu)2^{u+s}$, $a(i0) = 1$, $a(iu) = 0$ or 1, and $1 \leq i \leq e'$.

Let $r = 2^{s-1} + b_1 2^s + \ldots$. Then Lemma 7 implies that $C_i(2r)$ is even if $m(i) \notin I_s$ and hence Corollary 4 gives

$$\sum_{i \leq e'} d(i)C_i(2r) = 0 \bmod 2.$$

Since $\beta(m(i)/2r) = 0$, this gives

$$\sum d(i)\binom{m(i)+2r-1}{4r-1} = 0 \bmod 2.$$

From Lemma 6 we have

$$\binom{m(i)+2r-1}{4r-1} = \binom{b_2 + a(i2)}{b_1}\binom{b_3 + a(i3)}{b_2}\cdots.$$

Choose $b_i = 0$ for each i. Then all binomial coefficients in the above line become 1. Hence

$$\sum_{1\le i\le e'} d(i) = 0 \bmod 2.$$

This proves (i) since the left-hand side is Card I_s.

(ii) We can assume that the distinct $m(i)$ in I_s with odd $d(\ )$ are the first $2e$ from $\{m(j)\}$. From the proof of (i), it is clear that since we are assuming the $d(i)$ to be odd, the information we have is

$$(**)\qquad \sum_{i=1}^{2e} a(ik_1)\cdots a(ik_v) = 0 \bmod 2, \qquad v \ge 1, \quad 2 \le k_1 < \cdots < k_v.$$

When $e = 1$, take $v = 1$ in (**) to get $a(1u) = a(2u)$, $u > 1$. To "solve" (**) in general we need

Lemma 9. *Consider the following system over* $\mathbf{Z}_2$:

$$(**)\qquad \sum_{i=1}^{2e} a(i, k_1)\cdots a(i, k_v) = 0, \qquad 2 \le k_1 < \cdots < k_v.$$

This system is satisfied $\Longleftrightarrow$ the $a(\ ,\ k)$ are equal in pairs i.e. for each i, there is an i', $i \ne i'$, such that $a(i, k) = a(i', k)$ for all $k \ge 2$.

Proof. $\Longleftarrow$ Obviously the system is satisfied if $a(i, k) = a(i', k)$.

$\Longrightarrow$ Conversely, we solve (**) by induction on e. The conclusion of the lemma is true for $e = 1$. Let the conclusion be true for systems

$$\sum_{i=1}^{2e''} a'(i, k_1)\cdots a'(i, k_v) = 0, \qquad e > e'', \qquad 2 \le k_1 < \cdots < k_v.$$

If in (**) the a's are all 0 or all 1, we are finished. Assume therefore that the $a(i, 2)$ are not all equal. Clearly we can assume without loss of generality that

$$a(1, 2) = \cdots = a(2q, 2) = 1, \quad a(2q+1, 2) = \cdots = a(2e, 2) = 0$$

for some $q < e$.

In (**) take $k_1 = 2$. We get

$$\sum_{1\le i\le 2q} a(i, k_2)\cdots a(i, k_v) = 0, \qquad 3 \le k_2 < \cdots < k_v.$$

By the induction hypothesis, for each i, there is an i' $(1 \leq i, i' \leq 2q)$ such that $a(i, k) = a(i', k)$, $k \geq 3$. Hence from (**) we get

$$\sum_{2q+1 \leq i \leq 2e} a(i, k_2) \cdots a(i, k_v) = 0,$$

and by the induction hypothesis, for each i, there is an i' $(2q + 1 \leq i, i' \leq 2e)$ such that $a(i, k) = a(i', k)$, $k \geq 3$. This completes the proof of Lemma 9.

To complete the proof of Proposition 8(ii), take

$$b_i = a(i1), \qquad 1 \leq i \leq 2e \quad \text{and} \quad w_j = \sum_{u \geq 2} a(2j - 1, u)2^{u+s}, \qquad 1 \leq j \leq e.$$

Finally, we have our 2-primary result on $\{m(j)\}$.

Theorem. *With the notation of Proposition* 8, *each element of* I_s, $s > 0$, *has an even* $d(\ )$, *i.e. each element occurs an even number of times.*

Proof. This is an easy corollary of Proposition 8, for $f^!\psi^3 y_1$ is symplectic [2, p. 71]. Thus the argument of Proposition 8 gives

$$3m(2i - 1) = 2^s(1 + 4w'_i + 2b'_{2i-1}),$$

$$3m(2i) = 2^s(1 + 4w'_i + 2b'_{2i}) \quad \text{for some } w'_i \text{ and } b'_i.$$

Thus $m(2i) = m(2i - 1)$. Hence $e^* = 0$.

In summary, our 2-primary restriction is that in $\{m(j)\}$, each even $m(j)$ occurs an even number of times.

Corollary. *If all the* $m(i)^2$ *are equal, to* m^2 *say, then* n *odd* $\Longrightarrow m$ *odd or zero.*

Proof. Let $m \in I_s$. If $s > 0$, then Card $I_s = n$ is even by the Theorem.

Notes. (1) The case $n = 1$ is given in [6].

(2) It is clear from the Theorem that using our method, KSp will not give further information on $\{m(j)\}$.

REFERENCES

1. J. F. Adams, *Vector fields on spheres*, Ann. of Math. (2) 75 (1962), 603–632. MR 25 #2614.

2. ———, *Lectures on Lie groups*, Benjamin, New York, 1969. MR 40 #5780.

3. J. F. Adams and Z. Mahmud, *Classification of maps* $BG \to BG'$ (in preparation).

4. M. F. Atiyah and F. Hirzebruch, *Vector bundles and homogeneous spaces*, Proc. Sympos. Pure Math., vol. 3, Amer. Math. Soc., Providence, R.I., 1961, pp. 7–38. MR 25 #2617.

5. A. Borel, *Topics in the homology theory of fibre bundles*, Lecture Notes in Math., no. 36, Springer-Verlag, Berlin and New Yotk, 1967. MR 36 #4559.

6. *Conference on algebraic topology*, University of Illinois at Chicago Circle

(1968), Univ. of Illinois at Chicago Circle, Chicago, Ill., 1969, p. 293 (Conjecture 38). MR 40 #2050.

7. H. Samelson, *Notes on Lie algebras*, Van Nostrand Reinhold Math. Studies, no. 23, Van Nostrand Reinhold, New York, 1969, p. 144. MR 40 #7322.

8. N. E. Steenrod, *Milgram's classifying space of a topological group*, Topology 7 (1968), 349–368. MR 38 #1675.

DEPARTMENT OF PURE MATHEMATICS, THE UNIVERSITY, LIVERPOOL L69 3BX, ENGLAND

Current address: 7, Crediton Close, Manchester M15 6EW, England