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CLASSIFICATION OF HOMOTOPY TORUS KNOT SPACES

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ABSTRACT. The existence of nontrivial homotopy torus knot spaces is established as a corollary to the

Theorem. *Let* p *and* q *be two integers with* $p > 1$, $q > 1$, *and* $(p, q) = 1$. *Let* $\mathfrak{M}$ *be a maximal set of topologically distinct compact orientable irreducible* 3*-mainfolds with fundamental group presented by* $\langle a, b | a^p b^q \rangle$. *Then* $\operatorname{card}(\mathfrak{M}) = \frac{1}{2}\Phi(pq)$, *where* Φ *denotes Euler's function.*

All spaces are piecewise linear. The symbols I, D, and B denote the closed unit interval, disc, and ball, respectively. S^i denotes the i-dimensional sphere; $i = 1, 2, 3$. The closure and boundary of a space X are denoted respectively by $\mathrm{cl}(X)$ and ∂X. The term *knot space* refers to the closure of the complement in S^3 of a regular neighborhood of a knot. If m and n are positive integers and $(m, n) = 1$, then the torus knot space corresponding to the pair (m, n) is denoted by $Kt(m, n)$.

A 3-mainfold is *irreducible* if every embedded 2-sphere bounds a 3-cell. Let M be a compact manifold with boundary and K be a 2-complex in M. If $M - K$ is homeomorphic to $\partial M \times (0, 1]$, then K is called a *spine* of M. If M has empty boundary and K is a spine of $\mathrm{cl}(M - B)$, then we will say that K is a spine of M.

If M_1 and M_2 are compact 3-manifolds with boundary, $M_1 \subset M_2$, $\mathrm{cl}(M_2 - M_1) = U = D \times I$, and $M_1 \cap U = \partial D \times I$, then we will say that M_2 is obtained from M_1 *by attaching the* 2*-handle* U to M_1. If γ is a simple closed curve in ∂M_1 having $\partial D \times I$ as a regular neighborhood in ∂M_1, then we will say that U was attached to M_1 *along* γ.

We distinguish between the terms *group* and *group presentation.* If $\psi = \langle x_1, \ldots, x_m | R_1, \ldots, R_n \rangle$ is a group presentation, then G_ψ denotes the group presented by ψ, K_ψ the 2-complex corresponding to ψ, and P_ψ the corresponding P-graph (see [4]). Note that if K_ψ is a spine of the compact 3-manifold M, then $\pi_1(M) = G_\psi$.

Henceforth ψ denotes the presentation $\langle a, b | a^p b^q \rangle$.

Lemma. *Under the conditions of the Theorem, if* $M \in \mathfrak{M}$ *then* K_ψ *is a spine of* M.

Proof. By results of Waldhausen [5], M is a Seifert fiber space with

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orientable quotient surface E, and $\pi_1(M)$ is presented by

$$\psi^* = \langle x_1, \dots, x_{2g+r}, y_1, \dots, y_s, z \mid [x_i, z] = 1, y_i^{\mu_i} = z\rangle,$$

where the genus of E is $g \geqq 0$, where ∂M has $r+1>0$ components, each component a torus $S^1 \times S^1$, and where M has $s \geqq 0$ exceptional fibers with respective orders $\mu_1, \dots, \mu_s$. Since G_ψ is isomorphic to G_{ψ^*}, we infer that $r = g = 0$, $s = 2$, and that μ_1 and μ_2 are p and q in some order. This is argued by considering $\pi_1(M)$ modulo its center and applying results of [2, §4.1].

It now follows that E is a disc with two exceptional points x_1 and x_2 corresponding to the two exceptional fibers. Let γ be an arc in E with $\partial\gamma = \gamma \cap \partial E$ and γ separating x_1 from x_2. Let F be the union of all fibers projecting onto γ. Then F is an annulus that separates M into two solid tori T_1 and T_2. Write $F = S^1 \times I$ and consider the arc $\delta = \{t\} \times I$ in M, where $t \in S^1$. Note that $\partial\delta = \delta \cap \partial M$. Let U be a regular neighborhood of δ. Then $\mathrm{cl}\,(M - U)$ is a genus 2 handlebody to which the 2-handle U is attached. Clearly $U \cap T_1$ and $U \cap T_2$ are each connected. Thus U is attached to T according to the word $a^p b^q$, and the Lemma follows

Proof of the Theorem. Choose r and s so that $(r, p) = 1 \leqq r < p$ and $(s, q) = 1 \leqq s < q$. By the results of [3], we know that a compact orientable 3-manifold $M_{p,r,q,s}$ with boundary and with spine K_ψ is uniquely determined by the faithful embedding of P_ψ in S^2 with gap r on the a-syllable graph and gap s on the b-syllable graph. Any such manifold, having K_ψ as a spine, is thus of the same homotopy type as $Kt(p, q)$. It will suffice to show that $M_{p,r,q,s}$ and $M_{p,r',q,s'}$ are homeomorphic if and only if $r \equiv \epsilon r'$ (mod p) and $s \equiv \epsilon s'$ (mod q) where $\epsilon = \pm 1$.

The if part is clear; if $\epsilon = -1$, then we merely reverse the orientation.

Let T denote a genus 2 handlebody with inner meridian discs corresponding to the generators a and b. Following the construction of $M = M_{p,r,q,s}$, the above-mentioned faithful embedding of P_ψ determines a simple closed curve γ_0 (corresponding to the word $a^p b^q$) in ∂T. Then M is obtained from T by attaching a 2-handle U along γ_0, and ∂M is homeomorphic to the torus $S^1 \times S^1$.

One can construct simple closed curves γ_1 and γ_2 in $\partial T - U$ that correspond respectively to the words a^p and $a^r b^s$. Moreover this can be done so that $\partial M - (\gamma_1 \cup \gamma_2)$ is connected, γ_1 and γ_2 intersecting in a single crossing point.

To see this, we use the techniques of [3]. Construct in S^2 a faithfully embedded a-syllable graph with three syllables whose exponents are p, p, and r, respectively, and a faithfully embedded b-syllable graph with two syllables whose exponents are q and s. Note that this is possible since the gaps on the syllables a^p and b^q are r and s respectively. See Figure 1 for the construction. The ends of the a-syllables are indicated by 0, 1, and 2, and those of the b-syllables by 0 and 2. Connect the syllable ends with arcs as shown. In constructing T, we obtain simple closed curves γ_0, γ_1, and γ_2 in ∂T corresponding, respectively, to the words $a^p b^q$, a^p, and $a^r b^s$. Moreover γ_1 and γ_2 intersect in the point Q, which is clearly a crossing point. Attaching the 2-handle U to T along γ_0 gives us the manifold M with the curves γ_1 and γ_2 in $\partial M - U$ and intersecting at Q.

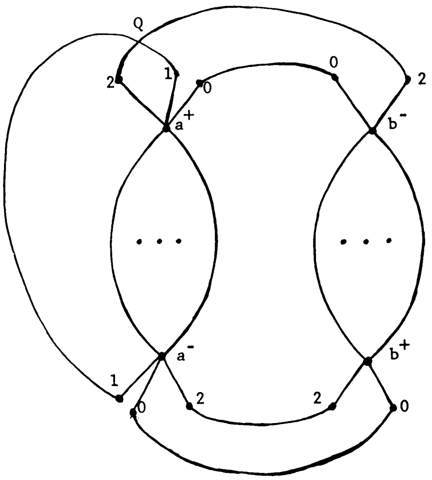

Figure 1

We show that $\gamma_1 \cup \gamma_2$ does not separate ∂M. Let

$$\psi_1 = \langle a, b \mid a^p b^q, a^p \rangle \quad \text{and} \quad \psi_2 = \langle a, b \mid a^p b^q, a^r b^s \rangle.$$

Then K_{ψ_1} and K_{ψ_2} are spines of closed manifolds—the former of a connected sum of two lens spaces (by the multiplication theorem [4]) and the latter of a lens space (see [4]). This means that each M_i has a 2-sphere boundary; hence, each curve γ_i does not separate ∂M ($i = 1, 2$). It follows that $\gamma_1 \cup \gamma_2$ does not separate ∂M.

Let γ be any nonseparating simple closed curve in ∂M. Since $\pi_1(\partial M)$ is abelian and is generated by a^p and $a^r b^s$, we observe that γ corresponds to the word $W = (a^p)^m (a^r b^s)^n$ in $\pi_1(M)$ for an appropriate choice of m and n ($(m, n) = 1$). Let $\widetilde{M}$ be a closed manifold obtained from M by attaching a 2-handle U to M along γ and then attaching a 3-cell to the 2-sphere boundary of the resulting manifold. Then $\pi_1(\widetilde{M})$ is presented by $\widetilde{\psi} = \langle a, b \mid a^p b^q, W \rangle$.

We show that $\widetilde{M}$ is a lens space if and only if $|n| = 1$. If $n = 0$, then $(m, n) = 1$ forces $m = \pm 1$ and $\widetilde{M}$ is a connected sum of two nontrivial lens spaces. If $|n| > 1$, then $\pi_1(\widetilde{M})$ has a homomorphism onto the group presented by $\langle a, b | a^p = b^q = (a^r b^s)^n = 1\rangle$. This group can be shown not to be cyclic (see [1, p. 71]).

If $|n| = 1$, we assume $n = 1$ and obtain $\tilde{\psi} = \langle a, b | a^p b^q, a^{mp+r} b^s\rangle$ with $K_{\tilde{\psi}}$ a spine of $\widetilde{M}$. Thus, $\widetilde{M}$ is a lens space. Let λ be the order of $\pi_1(\widetilde{M})$. Then $\lambda = |ps - q(mp + r)|$. Thus, assuming that $M_{p,r,q,s}$ and $M_{p,r',q,s'}$ are homeomorphic, we conclude that

$$ps - qr - mpq = \epsilon(ps' - qr' - m'pq)$$

for $\epsilon = \pm 1$ and appropriate choices of m and m'. Hence,

$$p(s - \epsilon s') - q(r - \epsilon r') = pq(m - \epsilon m'),$$

and the Theorem follows.

Corollary. *There exists a compact orientable irreducible 3-mainfold which is not embeddable in S^3 but which is of the same homotopy type as a torus knot space.*

Proof. Suppose that $M = M_{p,r,q,s}$ is embeddable in S^3. Then ∂M is a torus $S^1 \times S^1$ in S^3. Since M is not a solid torus, it follows that $\mathrm{cl}(S^3 - M)$ is a solid torus. Hence S^3 is obtainable from M by attaching a 2-handle along some nonseparating simple closed curve in ∂M and then attaching a 3-cell to the resulting 2-sphere boundary. Thus $ps - qr - mpq = \pm 1$ for some m, a condition that is violated for $p = 5$, $q = 3$, $r = 1$, and $s = 1$.

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