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ON LIKELIHOOD RATIOS OF MEASURES GIVEN BY MARKOV CHAINS

R. LePAGE AND V. MANDREKAR[1]

ABSTRACT. In this note we study conditions for absolute continuity and singularity of two measures induced by finite Markov chains. These conditions are derived from a general result on singularity-continuity dichotomy.

1. In [3] we considered, for general measures, the relation between equivalence-singularity dichotomies and zero-one laws with respect to the tail field of transition densities. In [4] Lodkin studied, under the exact analogue of Kolmogorov's zero-one law, the problem of the existence of the densities for two Markov chain measures on the space of real sequences. Unfortunately, the proof of Lodkin [4, Theorem 2], on which this theorem is based, is incomplete. In addition, Lodkin's conditions for the Markov case [4, Theorem 3] fail to cover the special case of the products of discrete measures, which was completely solved by Kakutani [1].

We first state a slight generalization of our dichotomy result for probability measures [3], which includes Lodkin [4, Theorem 2]. From this, sufficient conditions are derived for absolute continuity of one Markov chain measure with respect to another. These conditions improve those of Lodkin [4, Theorem 3]. Specializing further, we obtain conditions necessary and sufficient for absolute continuity of one Markov chain measure with respect to another, when a particular Condition A is satisfied. A large class of pairs of Markov chain measures is shown to satisfy Condition A. In particular, pairs of discrete product measures satisfy Condition A, and for this case the n.a.s. conditions we obtain for absolute continuity are exact analogues of those obtained by Kakutani [1, Corollary 1] in the Bernoulli case.

1.1. **Theorem.** *Let $\mathfrak{F}_1 \subset \mathfrak{F}_2 \subset \cdots$ be σ-algebras of subsets of Ω, $\mathfrak{F} = \sigma(\bigcup_n \mathfrak{F}_n)$ the σ-algebra generated by their union. For any probability measures P, Q on $\mathfrak{F}$ with Q absolutely continuous with respect to P ($Q \ll P$) when restricted to each of $\mathfrak{F}_1, \mathfrak{F}_2, \dots$, we let $\rho_n = (dQ|\mathfrak{F}_n)/(dP|\mathfrak{F}_n)$ and define the tail σ-algebra $\mathcal{G} = \bigcap_n \sigma\{\rho_{k+1}/\rho_k \mid k \geq n\}$. The following is then true: ($A \in \mathcal{G} \Rightarrow Q(A) = 0$ or 1 i.e. $\mathcal{G}$ is Q trivial) $\Rightarrow P \perp Q$ or $Q \ll P$ on*

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$\mathfrak{F}$. *Furthermore, if* $\mathcal{G}$ *is* Q*-trivial, we have* $Q \ll P$ *on* $\mathfrak{F}$ *iff* $E_P\rho_n^{1/2} \not\to 0$. $\square$

The first part of this theorem is proved as is the theorem in [3]. The second part is readily derived from the work of C. Kraft [2].

1.2. **Remark.** The error in [4] occurs since the author asserts [4, p. 692] that if P, Q are two measures such that $\partial Q/\partial P$ (the Radon-Nykodym derivative of the absolutely continuous part of Q with respect to P) satisfies $Q(\partial Q/\partial P > 0) = 1$, then $Q \ll P$. We caution that the Radon-Nykodym derivative $\partial Q/\partial P$ is defined a.e. P. Consider now $\Omega = \{0, 1\}$, $\mathfrak{F}$ = the σ-field all subsets of Ω. Define $Q\{i\} = 1/2$ $(i = 0, 1)$ and $P\{0\} = 1$, $P\{1\} = 0$. Then f defined by $f(0) = 1/2$, $f(1) = +\infty$ is a version of $\partial Q/\partial P$ (in some respects the most natural), and $Q(f > 0) = 1$. But Q is not absolutely continuous with respect to P. This difficulty was specifically avoided in [3].

Now, in view of Theorem 1.1, it suffices to obtain conditions for $E_P\rho_n^{1/2} \not\to 0$ once we assume that our zero-one law applies with respect to Q. Throughout the remainder we shall assume $\mathcal{G}$ is Q-trivial.

2. Let $\Omega_k = \{1, 2, \ldots, N\}$, $\Omega = \Pi_k \Omega_k$ and P, Q be two Markov measures given by the respective starting vectors $\{p_0(i)\}_{i=1}^N$, $\{q_0(i)\}_{i=1}^N$, and matrices of transitions $\{p_n(i, j)\}_{i,j=1}^N$, $\{q_n(i,j)\}_{i,j=1}^N$, $n = 1, 2, \ldots$. Let P_n and Q_n $(n = 1, 2, \ldots)$ be the corresponding Markov measures on the σ-algebra $\mathfrak{F}_n$ of cylinder sets with base a subset of $\Pi_{k=1}^n \Omega_k$. Suppose $Q_n \ll P_n$ and $\rho_n = dQ_n/dP_n$, $\forall n \geq 1$. In general, we know that

$$E_P\rho_n^{1/2} \not\to 0 \text{ iff } \prod_n \frac{E_P\rho_{n+1}^{1/2}}{E_P\rho_n^{1/2}} \text{ converges iff } \sum_n \left(1 - \frac{E_P\rho_{n+1}^{1/2}}{E_P\rho_n^{1/2}}\right) < \infty. \tag{2.1}$$

Hence under our zero-one law the condition $\Sigma_n(1 - E_P\rho_{n+1}^{1/2}/E_P\rho_n^{1/2}) < \infty$ is necessary and sufficient for the absolute continuity of Q with respect to P on $\mathfrak{F} = \sigma(\bigcup_n \mathfrak{F}_n)$.

For Markov measures we note that [4, p. 693] letting

$$a_{x_n}^n = \sum_{x_1 \cdots x_{n-1}} \sqrt{p_0(x_1)q_0(x_1) \cdots p_{n-1}(x_{n-1}, x_n)q_{n-1}(x_{n-1}, x_n)}$$

and $I_n = E_P\rho_n^{1/2}$, we have

$$\frac{I_{n+1}}{I_n} = \sum_{x_n} a_{x_n}^n \left\{\sum_{j=1}^N \sqrt{p_n(x_{n,j})}\sqrt{q_n(x_{n,j})}\right\} \Big/ \sum_{x_n} a_{x_n}^n. \tag{2.2}$$

Hence

$$\left(1 - \frac{I_{n+1}}{I_n}\right) = \sum_i \frac{a_i^n}{\Sigma_k a_k^n} \sum_{j=1}^N (\sqrt{p_n(i, j)} - \sqrt{q_n(i, j)})^2. \tag{2.3}$$

From equation (2.3) the following proposition is obvious.

2.4. **Proposition.**

$$\sum_n \left\{ \sum_{j=1}^N (\sqrt{p_n(i, j)} - \sqrt{q_n(i, j)})^2 \right\} < \infty \ \forall i \Rightarrow \sum_n \left(1 - \frac{E_P \rho_{n+1}^{1/2}}{E_P \rho_n^{1/2}} \right) < \infty. \quad \square$$

Since $(\sqrt{p_n(i, j)} - \sqrt{q_n(i, j)})^2 \le (p_n(i, j) - q_n(i, j))^2 / p_n(i, j)$ for all i, j, the above proposition gives Theorem 3 of [4].

We now specialize to a class of pairs of Markov chain measures for which the conditions above take simpler form.

Condition A. There exists $\delta > 0$,

$$E_P^{x_{n-1}} \sqrt{q_{n-1}(x_{n-1}, x_n)/p_{n-1}(x_{n-1}, x_n)}\, g_n(x_n) \ge \delta \sum_{i=1}^N g_n(i)$$

for sufficiently large n a.e. $[P]$, where $g_n(x_n) = \Sigma_j (\sqrt{p_n(x_n, j)} - \sqrt{q_n(x_n, j)})^2$.

The following theorem gives necessary and sufficient conditions for existence of densities subject to A.

2.5. **Theorem.** *Let P and Q be Markov chain measures on Ω such that* (i) $Q_n \ll P_n$, (ii) $\mathcal{G}$ *is Q-trivial, and* (iii) *Condition* A *is satisfied. Then* $\Sigma_n^\infty (\sqrt{p_n(i, j)} - \sqrt{q_n(i, j)})^2 < \infty$ *for all $i, j = 1, 2, \dots, N$ iff $Q \ll P$ on $\mathcal{F}$, otherwise $Q \perp P$ on $\mathcal{F}$.*

Proof. In view of Proposition 2.4 it suffices to prove that $Q \ll P$ implies $\Sigma_n (\sqrt{p_n(i, j)} - \sqrt{q_n(i, j)})^2 < \infty$ for all $i, j = 1, 2, \dots, N$. We first note the numerator of the right side of (2.3) can be written as

$$(2.6) \qquad E_P \sqrt{\frac{q_0(x_1)}{p_0(x_1)}}\, E^{x_1} \sqrt{\frac{q_1(x_1, x_2)}{p_1(x_1, x_2)}} \cdots E^{x_{n-1}} \sqrt{\frac{q_{n-1}(x_{n-1}, x_n)}{p_{n-1}(x_{n-1}, x_n)}}\, g_n(x_n)$$

where E^{x_i} denotes the conditional expectation given x_i, and $g_n(x_n) = \Sigma_j (\sqrt{p_n(x_n, j)} - \sqrt{q_n(x_n, j)})^2$. From (2.6) and Condition A we get

$$\left(1 - \frac{I_{n+1}}{I_n} \right) \ge \delta \frac{I_{n-1}}{I_n} \sum_{i=1}^N g_n(i).$$

If the sum on the right side of (2.1) is finite (i.e. if $Q \ll P$), then there is an M such that for $n \ge M$ we get $I_{n-1}/I_n > 1/2$. From the above we have that for $n \ge M$, $(1 - I_{n+1}/I_n) \ge \frac{1}{2} \delta \Sigma_{i=1}^N g_n(i)$. This implies the result. $\square$

2.7. **Remark.** (i) The above proof also shows that if P and Q obey Condition A and $Q \ll P$, then $\Sigma_n (\sqrt{p_n(i, j)} - \sqrt{q_n(i, j)})^2 < \infty$ for all i, j. (ii) If P and Q are Markov chain measures on Ω with $\mathcal{G}$ Q-trivial and if $\{p_0(i)\}_{i=1}^N, \{q_0(i)\}_{i=1}^N, \{p_n(i, j)\}_{i,j=1}^N, \{q_n(i, j)\}_{i,j=1}^N$ $(n = 1, 2, \dots)$ are bounded away from zero then

$$E^{x_{n-1}}\sqrt{\frac{q_n(x_{n-1}, x_n)}{p_n(x_{n-1}, x_n)}}g_n(x_n) \geq \sqrt{\delta}\sum_{i=1}^{N}\sqrt{p_{n-1}(x_{n-1}, i)}g_n(i) \geq \delta \sum_{i=1}^{N} g_n(i).$$

Hence Condition A is satisfied for measures in this class, and Theorem 2.5 therefore furnishes necessary and sufficient conditions for $Q \ll P$ on $\mathfrak{F}$.

We now derive the results of Kakutani where the measures P and Q on Ω are product measures. We note that in this case $p_n(i, j) = p_n(j)$ independent of i. Also $I_{n+1}/I_n = \Sigma_{j=1}^N \sqrt{p_n(j)}\sqrt{q_n(j)}$ and

$$E^{x_{n-1}}\sqrt{\frac{q_{n-1}(x_{n-1}, x_n)}{p_n(x_{n-1}, x_n)}}g_n(x_n) = E_P\sqrt{\frac{q_{n-1}(x_n)}{p_{n-1}(x_n)}}g_n(x_n). \tag{2.8}$$

We note that $g_n(x_n) = \Sigma_{j=1}^N (\sqrt{p_n(j)} - \sqrt{q_n(j)})^2$ is independent of x_n, and (2.8) is therefore equal to $\Sigma_{i=1}^N \sqrt{q_{n-1}(i)}\sqrt{p_{n-1}(i)}\,\Sigma_{j=1}^N (\sqrt{p_n(j)} - \sqrt{q_n(j)})^2$. If $Q \ll P$ then from (2.1), for $\delta > 0$, $\exists N_\delta$ such that $\Sigma_{i=1}^N \sqrt{q_{n-1}(i)}\sqrt{p_{n-1}(i)} > \delta$ for $n \geq N_\delta$. Hence $Q \ll P$ implies that P and Q satisfy Condition A. From Proposition 2.4 and Remark 2.7(i) we get the following which includes Kakutani [1, Corollary 1] since for product measures $\mathcal{G}$ is always Q-trivial [3].

2.9. **Corollary.** *If P and Q are product measures on Ω as above such that $Q_n \ll P_n$, then $Q \ll P$ on $\mathfrak{F}$ iff $\Sigma_n (\sqrt{p_n(i)} - \sqrt{q_n(i)})^2 < \infty$ for $i = 1, 2, \ldots, N$.*

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DEPARTMENT OF STATISTICS AND PROBABILITY, MICHIGAN STATE UNIVERSITY, EAST LANSING, MICHIGAN 48823