PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 52, October 1975

FINITE UNIONS OF IDEALS AND MODULES

PHILIP QUARTARARO, JR.[1] AND H. S. BUTTS

ABSTRACT. We say that a commutative ring R is a u-ring provided R has the property that an ideal contained in a finite union of ideals must be contained in one of those ideals; and a um-ring is a ring R with the property that an R-module which is equal to a finite union of submodules must be equal to one of them. The primary purpose of this paper is to characterize u-rings and um-rings. We show that R is a um-ring if and only if the residue field R/P is infinite for each maximal ideal P of R; and R is a u-ring if and only if for each maximal ideal P of R either the residue field R/P is infinite or the quotient ring R_P is a Bézout ring.

Introduction. All rings considered are commutative with identity $1 \neq 0$, and R denotes a ring with total quotient ring $T(R)$. For a multiplicative system S of R and a unitary R-module M, we use standard conventions concerning properties and notation for the quotient ring R_S and the quotient module M_S (e.g. see [G, p. 52], [N, p. 14], [K, p. 22], [ZS, p. 221]; in particular, AR_S will denote the ideal in R_S associated with the ideal A of R by the canonical map from R to R_S. The "running indices" will usually be dropped from finite intersections, unions, and sums; thus $\bigcup A_i$, $\bigcap A_i$, and ΣA_i will uniformly mean that i ranges from $i = 1$ to $i = n$. An ideal A of R is called a u-ideal provided $A \subset \bigcup A_i$ implies $A \subset A_i$ for some i, where $A_1, \ldots, A_n$ are ideals of R.

It is well known that an ideal of R contained in a finite union of prime ideals of R must be contained in one of those primes [K, p. 55], [M], but it does not seem to have been generally observed that this property holds in certain rings (e.g. Dedekind domains) even if none of the ideals involved are prime. In [M], McCoy shows that $A \subset \bigcup A_i$ implies $A^k \subset \bigcap A_i$ for some positive integer k in a general commutative ring, provided A is not contained in the union of any $n - 1$ of the A_i; in addition he proves an analogous result for subgroups of a group G.

1. Preliminary results. In this section we show that in considering u-rings we can replace $A \subset \bigcup A_i$ by $A = \bigcup A_i$, thus motivating the definition of um-rings; moreover, only finitely generated A need be considered. In addition, we show that every invertible ideal is a u-ideal.

Received by the editors July 11, 1974.
AMS (MOS) subject classifications (1970). Primary 13A15, 13F05, 13H99.
Key words and phrases. Bézout ring, quotient ring, residue field, quasi-local ring, Prüfer domain.

[1] This author was partially supported by NSF Grant GY 11075.

The proofs of the following two propositions are easy and we omit them.

Proposition 1.1. *The following conditions are equivalent, for* $A, A_1, \ldots, A_n$ *ideals of* R.

(1) *Each ideal of* R *is a u-ideal.*

(2) *Each finitely generated ideal of* R *is a u-ideal.*

(3) $A = \bigcup A_i$ *finitely generated* $\Rightarrow A = A_i$ *for some* i.

(4) $A = \bigcup A_i \Rightarrow A = A_i$ *for some* i.

Corollary 1.2. *Every Bézout ring (i.e., a ring in which every finitely generated ideal is principal) is a u-ring.*

Proposition 1.3. *If* R *is a u-ring (um-ring), then every homomorphic image and every quotient ring* R_S *of* R *is a u-ring (um-ring).*

It is clear that a *um*-ring is a *u*-ring, but the converse is false (e.g. the ring of integers). It is easy to check that if $M = \bigcup M_i$ implies $M = M_i$ for some i, where M is a finitely generated R-module and $M_1, \ldots, M_n$ are submodules of M, then R is a *um*-ring.

Proposition 1.4. *If* (0) *is the annihilator of the finitely generated ideal* A *of* R *and* $B_i \neq R$ *is an ideal of* R *for* $i = 1, \ldots, n$, *then* $A \not\subset \bigcup AB_i$.

Proof. We use induction on n. Consider $n = 1$ and suppose $A = AB_1$. There exists $b \in B_1$ such that $a = ab$ for all $a \in A$ [ZS, p. 215], implying $B_1 = R$, a contradiction. Suppose the proposition holds for $n - 1 \geq 1$ and consider two cases.

Case 1. Among $B_1, \ldots, B_n$ there are two ideals, say B_1 and B_2, such that $B_1 + B_2 \neq R$. Since $A \not\subset A(B_1 + B_2) \cup AB_3 \cup \ldots \cup AB_n$, it follows that $A \not\subset \bigcup AB_i$, completing Case 1. In Case 2, suppose $B_i + B_j = R$ for all $i \neq j$. Let $C_i = \Pi B_j$ $(j \neq i)$ for $i = 1, \ldots, n$. Then $B_i + C_i = R$ [ZS, p. 177] and $AB_i + AC_i = A$ for $i = 1, \ldots, n$. If $AC_i \subset AB_i$ for some i, then $A = AB_i + AC_i = AB_i$, which is impossible by the case $n = 1$. Hence there exists $a_i \in AC_i \setminus AB_i$ for $i = 1, \ldots, n$. Set $a = \Sigma a_i$ and note that $a \in A \setminus AB_i$ for $i = 1, \ldots, n$, completing the proof.

By a fractional ideal F of R we understand an R-module $F \subset T(R)$ for which there exists a regular element (i.e., not a zero divisor) r of R such that $rF \subset R$.

Theorem 1.5. *Every invertible ideal of* R *is a u-ideal.*

Proof. If A is an invertible ideal and $A_1, \ldots, A_n$ are ideals of R such that $A \subset \bigcup A_i$, then $A = \bigcup C_i$ where $C_i = A \cap A_i$. Suppose $A \neq C_i$ for $i = 1, \ldots, n$. Then $C_i = AB_i$ for each i, where $R \neq B_i = C_i A^{-1}$ is an ideal of R. Since A is invertible, the annihilator of A is (0) and A is finitely generated [ZS, p. 272]. Proposition 1.4 implies $A \not\subset \bigcup AB_i = \bigcup C_i$, a contradiction.

Corollary 1.6. *Every Prüfer domain is a u-domain.*

Griffin [MG] defines a Prüfer ring as a ring in which every finitely generated regular ideal is invertible; this type of Prüfer ring need not be a u-ring as can be seen by taking $R = Z[x]/(2, x)^2$, where $Z[x]$ is the polynomial ring over the integers. Butts and Smith [BS] called R a Prüfer ring provided the ideals of R_P are linearly ordered by inclusion for every proper prime ideal P of R; it follows from the characterization given in §2 that Prüfer rings of this type are u-rings.

Proposition 1.7. *If R contains an infinite set S such that $x - y$ is a unit of R for all $x \neq y$ in S, then R is a um-ring.*

Proof. Suppose $M_1, \ldots, M_n$ are submodules of an R-module M such that $M = \bigcup M_i$ and $M \neq M_i$ for each i. It is no restriction to assume for each i that $M_i \not\subset \bigcup M_j$ $(j \neq i)$, so let $m_i \in M_i$ and $m_i \notin \bigcup M_j$ $(j \neq i)$ for $i = 1, 2$ and consider the set $E = \{m_1 + xm_2 | x \in S\}$. There exist x and y in S such that $x \neq y$ and both $m_1 + xm_2$ and $m_1 + ym_2$ belong to the same M_i for some $i \neq 2$, implying $m_2 \in M_i$, a contradiction.

Theorem 1.8. *If there exists a maximal ideal P of R such that $F = R/P$ is finite with $n - 1$ elements, then for an R-module M (ideal M) such that the vector space $V = M/PM$ is not 1-dimensional over F there exist n submodules (ideals of R) $M_i \subset M$ such that $M \neq M_i$ for $i = 1, \ldots, n$ and $M = \bigcup M_i$.*

Proof. Let B be a basis for V over F and let $b_1 \neq b_2$ be two elements of B. Consider the following subsets of V; $E_1 = B\backslash\{b_1\}$, $E_2 = B\backslash\{b_2\}$, and $E_{2+i} = (B\backslash\{b_1, b_2\}) \cup \{b_1 + x_i b_2\}$ where x_i $(i = 1, \ldots, q - 1)$ ranges over the nonzero elements of F. Let S_j be the subspace of V generated by E_j for $j = 1, \ldots, n$. A routine argument shows that $V = \bigcup S_i$ and $S_i \neq V$ for $i = 1, \ldots, n$. Hence, if $f: M \to V$ is the canonical homomorphism and $M_i = f^{-1}(S_i)$ for $i = 1, \ldots, n$, $M = \bigcup M_i$ and $M \neq M_i$ for each i.

Corollary 1.9. *Let R be a quasi-local ring with maximal ideal P.*

(i) *If R is a um-ring, then R/P is infinite.*

(ii) *If R is a u-ring, then either R/P is infinite or R is a Bézout ring.*

Proof. If M is finitely generated and not principal, then so is M/PM as a vector space over R/P [N, p. 13]; moreover, $M = R \oplus R$ is such a module.

2. **Characterization of u-rings and um-rings.** We first reduce the general case to the case in which R has only finitely many maximal ideals, and then deal with that situation.

Proposition 2.1. *A ring R is a um-ring (u-ring) if and only if R_S is a um-ring (u-ring) for each multiplicative system S of R which is the complement of a finite union of maximal ideals of R.*

Proof. If R is a *um*-ring (*u*-ring), then so is R_S by Theorem 1.3. Now, consider the converse and suppose $M = \bigcup M_i$, where M_i is a proper submodule of the R-module M (ideal M of R) for $i = 1, \ldots, n$. Let $m_i \in M \backslash M_i$ and let P_i be a maximal ideal of R containing the ideal $A_i = [M_i : m_i]_R$ for $i = 1, \ldots, n$. If S denotes the complement of $\bigcup P_i$ in R, then $M_S = \bigcup (M_i)_S$ is clear and $M_S \neq (M_i)_S$ since $m_i \notin (M_i)_S$.

Theorem 2.2. *If R has only finitely many maximal ideals, say $M_1, \ldots, M_n$, then the following statements are equivalent.*

(a) *R is a um-ring.*

(b) *R_P is a um-ring for every maximal ideal P of R.*

(c) *R/P is infinite for every maximal ideal P of R.*

(d) *There exists an infinite set S in R such that $x - y$ is a unit in R for all $x \neq y$ in S.*

Proof. (a) $\Rightarrow$ (b) by Proposition 1.3, (b) $\Rightarrow$ (c) by Corollary 1.9, and (d) $\Rightarrow$ (a) by Proposition 1.7. Consider (c) $\Rightarrow$ (d). Let S_i be a complete set of representatives for the nonzero elements of R/M_i, and let $\{s_{ij} \mid j = 1, 2, \ldots, \infty\}$ be a denumerable subset of S_i for $i = 1, \ldots, n$. For each j, there exists x_j in R such that $x_j \equiv s_{ij} (\mathrm{mod}\ M_i)$ for $i = 1, \ldots, n$ [**ZS**, p. 177]. Denote by S the set of all x_j thus obtained and observe that for $x_r \neq x_s$ in S we have $x_r - x_s \notin \bigcup M_i$ and $x_r - x_s$ is a unit of R.

Theorem 2.3. *The following statements are equivalent.*

(a) *R is a um-ring.*

(b) *R_P is a um-ring for every maximal ideal P of R.*

(c) *R/P is infinite for every maximal ideal P of R.*

Proof. This follows directly from Propositions 1.3, 2.1, and Theorem 2.2.

Lemma 2.4. *If R has only finitely many maximal ideals and R_M is a Bézout ring for each maximal ideal M of R, then R is a Bézout ring.*

Proof. Let $M_1, \ldots, M_n$ be the maximal ideals of R and for each $i = 1, \ldots, n$, choose $m_i \in M_j$ for $j \neq i$ and $m_i \notin M_i$. For a finitely generated ideal A of R, there exists $a_i \in A$ such that $AR_{M_i} = a_i R_{M_i}$ for $i = 1, \ldots, n$. If $a = \Sigma a_i m_i$, then $AR_{M_i} = aR_{M_i}$ for $i = 1, \ldots, n$ since $a_j R_{M_i} \subset AR_{M_i} = a_i R_{M_i}$ implies "a_i divides a_j" in R_{M_i} and $aR_{M_i} = a_i u_i R_{M_i}$ where u_i is a unit in R_{M_i}. Consequently $A = aR$.

Theorem 2.5. *If R has only finitely many maximal ideals, say $M_1, \ldots, M_r$, and R_{M_i} is a Bézout ring for $i = 1, \ldots, n$ $(1 < n \leq r)$ while R/M_i is infinite for $i > n$, then R is a u-ring.*

Proof. We can assume $r > n$ since $r = n$ implies R is a Bézout ring by Lemma 2.4 and therefore a u-ring. Suppose $A = (a_1, \ldots, a_m)$ and $A_1, \ldots, A_s$

are ideals in R with $A = A_1 \cup \dots \cup A_s$. For $B = R\backslash \bigcup M_i$ and $E = R\backslash[\bigcup M_i\ (i > n)]$, R_B is a Bézout ring by Lemma 2.4 and R_E has the property that each maximal ideal has infinite residue field. Theorem 2.2 implies that there exists an infinite set S^* in R such that $x - y$ is a unit in R_E for all $x \neq y$ in S^*. Choose $m \in \bigcap M_i \backslash [\bigcup M_i\ (i > n)]$ and set $S = \{ms | s \in S^*\}$. Then S is an infinite set in R such that $S \subset \bigcap M_i$ and $x - y$ is a unit in R_E for all $x \neq y$ in S. Let $x_1, x_2, x_3, \dots, x_i, \dots$ be a denumerable subset of S. Since R_B is a Bézout ring, we have $AR_B = aR_B$ for some $a \in A$. Consider the expressions $b_i = a + \Sigma_{j=1}^m a_j x_i^j$ for $i = 1, 2, 3, \dots, \infty$. There must be $m + 1$ values of the index i such that b_i belongs to some one of the A_j, say $b_1, \dots, b_{m+1}$ belong to A_1. We claim that $AR_{M_i} = A_1 R_{M_i}$ for $i = 1, \dots, r$, and therefore $A = A_1$. We deal first with the case $i > n$. Solving the system of equations $b_i = a + \Sigma_{j=1}^m a_j x_i^j$ $(1 \leq i \leq m + 1)$ for the a_j by Cramer's rule, we see that $a_j d \in A_1$, where $d = \Pi(x_i - x_j)\ (i > j)$ since it is a Vandermonde determinant. Consequently $a_j R_E \subset A_1 R_E$ for $j = 1, \dots, m$ and $AR_E = A_1 R_E$; hence $AR_{M_i} = A_1 R_{M_i}$ for $i > n$. On the other hand, $b_1 R_B = aR_B$ since $a_i R_B \subset AR_B = aR_B$ for $i = 1, \dots, m$ implies "a divides a_i" in R_B and hence $b_1 R_B = a(1 + j)R_B$ where j is in the Jacobson radical of R_B. Hence $aR_B \subset A_1 R_B \subset AR_B = aR_B$, $AR_B = A_1 R_B$, and $AR_{M_i} = A_1 R_{M_i}$ for $i = 1, \dots, n$, completing the proof.

Theorem 2.6. *The following statements are equivalent.*

(a) *R is a u-ring.*

(b) *R_P is a u-ring for every maximal ideal P of R.*

(c) *For each maximal ideal P of R, either the residue field R/P is infinite or the quotient ring R_P is a Bézout ring.*

Proof. (a) $\Longrightarrow$ (b) by Proposition 1.3, (b) $\Longrightarrow$ (c) by Corollary 1.9, and (c) $\Longrightarrow$ (a) by Proposition 2.1 and Theorem 2.5.

3. Some applications and examples. Applying Proposition 1.7, it is clear that a unitary overring of an infinite field is a *um*-ring, and that the ring $R(x)$ is a *um*-ring for any ring R [N, p. 18], [G, p. 410]. The following results can be established by standard techniques using Theorems 2.3 and 2.6. A finite direct sum of rings is a *um*-ring (*u*-ring) if and only if each summand is; and if any one of R, $R[x]$, $R[[x]]$ is a *um*-ring, so are the other two. Let D be an integral domain with quotient field K. If D is a *u*-ring and $D \subset R \subset K$, then R is a *u*-domain; if D/P is finite for all maximal ideals P of D, then D is a *u*-domain if and only if D is a Prüfer domain. If F is a finite algebraic extension field of the rational numbers and $F \supset R \supset Z$, then R is a *u*-ring if and only if R is integrally closed. Let J be the integral closure of D in an algebraic extension field L of K; if D is a *u*-ring, then J is a *u-ring*; if J is a *u-ring*, $[L: K]$ is finite, and D is integrally closed, then D is a *u*-ring.

Of course, a finite ring R is a u-ring if and only if R is a principal ideal ring.

REFERENCES

[BS]. H. S. Butts and W. W. Smith, *Prüfer rings*, Math. Z. 95 (1967), 196–211. MR 35 #173.

[G]. R. W. Gilmer, Jr., *Multiplicative ideal theory*, Dekker , New York, 1972.

[MG]. M. Griffin, *Prüfer rings with zero divisors*, J. Reine Angew. Math. 239/240 (1969), 55–67. MR 41 #188.

[K]. I. Kaplansky, *Commutative rings*, Allyn and Bacon, Boston, Mass., 1970. MR 40 #7234.

[M]. N. H. McCoy, *A note on finite unions of ideals and subgroups*, Proc. Amer. Math. Soc. 8 (1957), 633–637. MR 19, 246.

[N]. M. Nagata, *Local rings*, Interscience Tracts in Pure and Appl. Math., no. 13, Interscience, New York, 1962. MR 27 #5790.

[ZS]. O. Zariski and P. Samuel, *Commutative algebra*. Vol. I, University Ser. in Higher Math., Van Nostrand, Princeton, N. J., 1958. MR 19, 833.

DEPARTMENT OF MATHEMATICS, SOUTHERN UNIVERSITY, BATON ROUGE, LOUISIANA 70813

DEPARTMENT OF MATHEMATICS, LOUISIANA STATE UNIVERSITY, BATON ROUGE, LOUISIANA 70803