UNIMAXIMAL ORDERS

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ABSTRACT. Let $R$ be a Dedekind domain with quotient field $K$, and let $A$ be a separable $K$-algebra. An $R$-order $\Lambda$ in $A$ is said to be unimaximal if $\Lambda$ is contained in a unique maximal $R$-order in $A$. Unimaximal orders are given characterizations which are applied to determine those finite groups $G$ of order $n$ for which $RG$ is unimaximal, where $K$ is an algebraic number field containing a primitive $n$th root of unity.

1. Introduction. We assume throughout this paper that $R$ is a Dedekind domain with quotient field $K$, and that $A$ is a separable $K$-algebra. An $R$-order $\Lambda$ in $A$ is an $R$-subalgebra of $A$ which is finitely generated as an $R$-module and which contains a $K$-basis for $A$. We will follow the notation and terminology of [5] concerning orders and lattices.

Definition. An $R$-order $\Lambda$ in $A$ is said to be unimaximal if $\Lambda$ is contained in a unique maximal $R$-order in $A$.

The principal tools in this paper are the characterizations of hereditary orders given by Brumer [1], Harada [2] and Jacobinski [3]. Hereditary orders correspond locally to certain subrings of rings of the type $\text{End}_D(M_D)$, where $D$ is a division ring and $M_D$ is a finite dimensional right $D$-module; more importantly, a nonmaximal hereditary order corresponds at some prime to a subring which acts reducibly on some $M$.

Definition. Let $C$ be a unital subring of a ring $B$. We say $C$ is an irreducible subring of $B$ if for all simple left $B$-modules $M$, $M$ has no proper $(C, D)$-bisubmodules, where $D = \text{End}_B(M)$ acts on the right of $M$.

The Jacobson radical of a ring $B$ is denoted $J(B)$. From the definition we deduce the following

Proposition. Let $C$ be an irreducible subring of $B$. If $(C + J(B))/J(B)$ is artinian, then $J(B) \supseteq J(C)$.

Proof. Set $J = J(B)$ and $\bar{C} = (C + J)/J$. If $M$ is a simple left $B$-module, then $M$ is a nonzero left $\bar{C}$-module. Since $\bar{C}$ is artinian, $M$ contains a simple left $\bar{C}$-submodule, say $N$. Setting $D = \text{End}_B(M)$, we find that $ND$ is a nonzero $(C, D)$-bisubmodule of $M$, and so by assumption $ND = M$. It follows that the annihilator of $M$ in $B$ contains the annihilator of $N$ in $C$, and therefore $J(B) \supseteq J(C)$.
2. Characterizations. Let $\text{spec}(R)$ denote the set of prime ideals of $R$. If $P \in \text{spec}(R)$ and if $X$ is a finitely generated $R$-module, we let $\hat{X}_p$ (or $\hat{X}$ if there is no ambiguity) denote the $P$-adic completion of $X$.

**Theorem A.** Let $\Lambda$ be an $R$-order in $\Lambda$, and let $\Gamma$ be a fixed maximal order containing $\Lambda$. Then the following statements are equivalent:

(a) $\Lambda$ is unimaximal.

(b) If $H$ is a hereditary $R$-order in $\Lambda$ such that $\Lambda \subseteq H$, then $H$ is maximal.

(c) If $H$ is a hereditary $R$-order in $\Lambda$ such that $\Lambda \subseteq H \subseteq \Gamma$, then $H = \Gamma$.

(d) $\hat{\Lambda}_p$ is an irreducible subring of $\hat{\Gamma}_p$ for all $P \in \text{spec}(R)$.

(e) If $\Lambda'$ is an $R$-order such that $\Lambda \subseteq \Lambda' \subseteq \Gamma$, then $\hat{\Lambda}'_p \supseteq \hat{\Gamma}_p$ for all $P \in \text{spec}(R)$.

**Proof.** The proof will follow this scheme:

$$
\begin{align*}
& a \implies b: \text{Assume (a), and let } H \text{ be a hereditary } R\text{-order in } \Lambda \text{ such that } \Lambda \subseteq H. \text{ By [3, Proposition 3], } H \text{ is the intersection of maximal orders. Since } \Lambda \text{ is unimaximal, } H \text{ must be maximal, as desired.}
\end{align*}
$$

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\begin{align*}
& b \implies c: \text{This is obvious.}
\end{align*}
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\begin{align*}
& c \implies d: \text{Assume (d) is false, and fix some } P \in \text{spec}(R) \text{ such that } \hat{\Lambda} \text{ is not irreducible in } \hat{\Gamma}. \text{ Let } K \text{ be the quotient field of } R, \text{ and set } \hat{\Lambda} = A \otimes_K \hat{K}, \text{ so that } \hat{\Lambda} \text{ and } \hat{\Gamma} \text{ are } \hat{R}\text{-orders in } \hat{\Lambda}. \text{ It is no loss to assume that } \hat{\Lambda} \text{ is simple. Since } \hat{\Gamma} \text{ is a maximal } \hat{R}\text{-order in the simple algebra } \hat{\Lambda} \text{ and } \hat{\Gamma} \text{ is complete, } \hat{\Gamma} \text{ has a unique simple module } M. \text{ By hypothesis, } M \text{ contains a proper } (\hat{\Lambda}, D)\text{-submodule, say } N, \text{ where } D \text{ is the endomorphism ring of } M \text{ over } \hat{\Gamma}. \text{ By [3, Proposition 2], } \hat{H} = \{x \in \hat{\Gamma}: xN \subseteq N\} \text{ is a nonmaximal hereditary order in } \hat{\Gamma}, \text{ and clearly } \Lambda \subseteq \hat{H}. \text{ It is now easy to construct a nonmaximal hereditary order } H \text{ such that } \Lambda \subseteq H \subseteq \Gamma. \text{ Thus (c) is false, as desired.}
\end{align*}
$$

$$
\begin{align*}
& d \implies a: \text{Let (d) hold. It is enough to assume that } \Lambda \text{ is simple and } R \text{ is complete. It follows that } J = J(\Gamma) \text{ is the unique maximal two-sided ideal of } \Gamma. \text{ Let } \Gamma' \text{ be a maximal order containing } \Lambda, \text{ and set } \Lambda' = \Lambda \cap \Gamma'. \text{ Clearly } \Lambda' \supseteq \Lambda. \text{ Since } R \text{ is complete, (d) implies that } \Lambda, \text{ and hence } \Lambda', \text{ is irreducible as a subring of } \Gamma, \text{ so it follows that } (\Lambda' + J)/J \text{ is simple. Since } J \text{ is an invertible } \Gamma\text{-ideal, there exists an integer } n \text{ such that } J^n \Gamma' \subseteq \Gamma \text{ but } J^n \Gamma' \not\subseteq J, \text{ and we set } I = J^n \Gamma'. \text{ Notice that } I \text{ is a left ideal of } \Gamma. \text{ Since } \Gamma' \text{ is a maximal order, the right order } O(I) \text{ of } I \text{ is } \Gamma'. \text{ But then } \Gamma \cap O(I) = \Gamma \cap \Gamma' = \Lambda', \text{ and therefore } I \text{ is a two-sided ideal in } \Lambda'. \text{ It follows that } (I + J)/J \text{ is a two-sided ideal in } (\Lambda' + J)/J, \text{ and since the latter is a simple ring, either } I + J = J \text{ or } I + J = \Lambda' + J. \text{ The first possibility is ruled out.}
\end{align*}
$$
since \( I \subseteq J \), and hence
\[
(*) \quad I + J = \Lambda' + J.
\]
Now \( I \) is a left ideal of \( \Lambda' + J \), and \( J \) is contained in the radical of \( \Lambda' + J \), so Nakayama’s lemma applied to \( (*) \) shows that \( I = \Lambda' + J \). In particular, \( 1 \in I \). Inasmuch as \( I \) is a left ideal of \( \Gamma \), we find that \( I = \Gamma \), whence \( \Gamma = O_*(I) = \Gamma' \), showing that \( (d) \) implies \( (a) \).

\( d \Rightarrow e \): This follows directly from the Proposition in §1.

\( e \Rightarrow c \): Let \( (e) \) hold, and assume \( H \) is a hereditary \( R \)-order such that \( \Lambda \subseteq H \subseteq \Gamma \). By assumption, \( J(\hat{H}_P) \subseteq J(\hat{\Gamma}_P) \) for all \( P \in \text{spec}(R) \). Jacobinski’s characterization [3, Theorem 1] of hereditary orders implies that \( H = \Gamma \), as desired.

This concludes the proof of the theorem.

It is interesting to single out the following special case:

**Corollary.** Let \( \Gamma \) and \( \Gamma' \) be distinct maximal \( R \)-orders in \( \Lambda \). Then \( \Gamma \cap \Gamma' \) is contained in a nonmaximal hereditary order.

**Example.** Let \( K \) be the field of rational numbers, and let \( Z_{(2)} \) be the localization of the ring \( Z \) of integers at the prime 2. The polynomial \( x^2 + x + 1 \) is irreducible over the residue class field \( Z/2Z \) of \( \hat{Z}_{(2)} \), and therefore if \( \Lambda \) is any \( Z_{(2)} \)-order in \( K_2 \) (the ring of two-by-two matrices over \( K \)) which contains the companion matrix \( \begin{pmatrix} 0 & 1 \\ -1 & -1 \end{pmatrix} \) of \( x^2 + x + 1 \), then \( \Lambda \) is unimaximal.

3. **Group algebras.** We use a theorem in modular character theory together with the result of the previous section to characterize those group algebras (with suitable restrictions on the ground ring) which are unimaximal.

A finite group \( G \) is said to be \( p \)-nilpotent (\( p \) a rational prime) if \( G \) contains a normal subgroup \( N \) of order relatively prime to \( p \) such that \( G/N \) is a \( p \)-group.

**Theorem B.** Let \( G \) be a finite group of order \( n \), let \( K \) be an algebraic number field containing a primitive \( n \)th root of unity, and let \( R \) be a Dedekind domain with quotient field \( K \). Then \( RG \) is unimaximal in \( KG \) if and only if

1. \( G \) is \( p \)-nilpotent, and
2. the Sylow \( p \)-subgroups of \( G \) are abelian for all rational primes \( p \) which are nonunits in \( R \).

**Proof.** Let \( \Gamma \) be a maximal \( R \)-order in \( KG \) which contains \( \Lambda = RG \), and fix \( P \in \text{spec}(R) \). Let \( p \) be the rational prime in \( P \). Since \( K \) is a splitting field for \( KG \), it follows that \( J(\hat{\Gamma}) = \hat{P}\hat{\Gamma} \). Now let \( M \) be an irreducible (in the sense of lattices) left \( \hat{\Gamma} \)-lattice, so that \( M \) is also an irreducible left \( \hat{\Lambda} \)-lattice (see [5, Chapter IV, Lemma 1.13]). One easily checks that \( M/\hat{P}M \) is a simple left \( \hat{\Lambda} \)-module, and every simple left \( \hat{\Lambda} \)-module can be obtained in this way. Clearly \( \text{End}_{\hat{\Lambda}}(M/\hat{P}M) = \hat{R}/\hat{P} \), so any \( (\hat{\Lambda}, \hat{R}/\hat{P}) \)-bisuremodule of \( M/\hat{P}M \) is
simply a left $\mathcal{A}$-submodule. From Theorem A, we see that $\mathcal{A}$ is unimaximal if and only if each such $M/\mathcal{P}M$ is a simple $\mathcal{A}$-module. Noting that $\mathcal{A}/\mathcal{P}\mathcal{A} = F\mathcal{G}$, where $F = \hat{R}/\mathcal{P}$, we can apply a theorem of Richen [4] which says that $M/\mathcal{P}M$ is a simple $\mathcal{A}$-module for all such $M$ if and only if $G$ is $p$-nilpotent and the Sylow $p$-subgroups are abelian. This concludes the proof.

**Corollary.** Let $G$ be a finite group of order $n$, let $K$ be an algebraic number field containing a primitive $n$th root of unity, and let $R$ be the ring of algebraic integers in $K$. Then $RG$ is unimaximal in $KG$ if and only if $G$ is abelian.

**Proof.** Note first that every rational prime $p$ is a nonunit in $R$. From Theorem B we see that $RG$ is unimaximal in $KG$ if and only if $G$ is $p$-nilpotent with abelian Sylow $p$-subgroups for all primes $p$. One can readily show that this is equivalent to $G$ being abelian.

**Example.** Let $G = S_3$, the (nonabelian) symmetric group on 3 letters, of order 6. Let $K$ be the field of rational numbers, and let $R = \mathbb{Z}$. Since $G$ is not 3-nilpotent and 3 is not a unit in $R$, $RG$ is not unimaximal. We give an explicit proper hereditary order containing $RG$. Write $KG = K \oplus K \oplus K_2$, a direct sum of full matrix algebras over $K$. Now $G$ is generated by the two 2-cycles $(12)$ and $(13)$. The correspondence

\[(12) \rightarrow \begin{pmatrix} 1 & -1 \\ 1 & -1 \end{pmatrix}, \quad (13) \rightarrow \begin{pmatrix} 1 & 3 \\ 0 & -1 \end{pmatrix}\]

defines an embedding of $RG$ into $KG$. Now the elements of $KG$ of the form \((a, b, c, d, e, f)\), where $a, b, c, d, e, f \in R$, is a proper hereditary order in $KG$ which clearly contains $RG$. From this the reader may determine two distinct maximal $R$-orders in $KG$ which contain $RG$.

**REFERENCES**


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