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UNIMAXIMAL ORDERS

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ABSTRACT. Let R be a Dedekind domain with quotient field K, and let A be a separable K-algebra. An R-order Λ in A is said to be unimaximal if Λ is contained in a unique maximal R-order in A. Unimaximal orders are given characterizations which are applied to determine those finite groups G of order n for which RG is unimaximal, where K is an algebraic number field containing a primitive nth root of unity.

1. **Introduction.** We assume throughout this paper that R is a Dedekind domain with quotient field K, and that A is a separable K-algebra. An R-order Λ in A is an R-subalgebra of A which is finitely generated as an R-module and which contains a K-basis for A. We will follow the notation and terminology of [5] concerning orders and lattices.

Definition. An R-order Λ in A is said to be unimaximal if Λ is contained in a unique maximal R-order in A.

The principal tools in this paper are the characterizations of hereditary orders given by Brumer [1], Harada [2] and Jacobinski [3]. Hereditary orders correspond locally to certain subrings of rings of the type $\mathrm{End}_D(M_D)$, where D is a division ring and M_D is a finite dimensional right D-module; more importantly, a nonmaximal hereditary order corresponds at some prime to a subring which acts reducibly on some M.

Definition. Let C be a unital subring of a ring B. We say C is an irreducible subring of B if for all simple left B-modules M, M has no proper (C, D)-bisubmodules, where $D = \mathrm{End}_B(M)^\circ$ acts on the right of M.

The Jacobson radical of a ring B is denoted $J(B)$. From the definition we deduce the following

Proposition. *Let C be an irreducible subring of B. If $(C + J(B))/J(B)$ is artinian, then $J(B) \supseteq J(C)$.*

Proof. Set $J = J(B)$ and $\overline{C} = (C + J)/J$. If M is a simple left B-module, then M is a nonzero left $\overline{C}$-module. Since $\overline{C}$ is artinian, M contains a simple left $\overline{C}$-submodule, say N. Setting $D = \mathrm{End}_B(M)^\circ$, we find that ND is a nonzero (C, D)-bisubmodule of M, and so by assumption $ND = M$. It follows that the annihilator of M in B contains the annihilator of N in C, and therefore $J(B) \supseteq J(C)$.

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2. **Characterizations.** Let spec(R) denote the set of prime ideals of R. If $P \in$ spec(R) and if X is a finitely generated R-module, we let $\hat{X}_P$ (or $\hat{X}$ if there is no ambiguity) denote the P-adic completion of X.

Theorem A. *Let Λ be an R-order in A, and let Γ be a fixed maximal order containing Λ. Then the following statements are equivalent:*

(a) *Λ is unimaximal.*

(b) *If H is a hereditary R-order in A such that $\Lambda \subseteq H$, then H is maximal.*

(c) *If H is a hereditary R-order in A such that $\Lambda \subseteq H \subseteq \Gamma$, then $H = \Gamma$.*

(d) *$\hat{\Lambda}_P$ is an irreducible subring of $\hat{\Gamma}_P$ for all $P \in$* spec(R).

(e) *If Λ' is an R-order such that $\Lambda \subseteq \Lambda' \subseteq \Gamma$, then $J(\hat{\Lambda}'_P) \supseteq J(\hat{\Gamma}_P)$ for all $P \in$* spec(R).

Proof. The proof will follow this scheme:

$$\begin{tikzcd} & d \arrow[r, Rightarrow] \arrow[dl, Rightarrow] & a \arrow[dd, Rightarrow] \\ e \arrow[dr, Rightarrow] & & \\ & c \arrow[uu, Rightarrow] & b \arrow[l, Rightarrow] \end{tikzcd}$$

a $\Rightarrow$ b: Assume (a), and let H be a hereditary R-order in A such that $\Lambda \subseteq H$. By [3, Proposition 3], H is the intersection of maximal orders. Since Λ is unimaximal, H must be maximal, as desired.

b $\Rightarrow$ c: This is obvious.

c $\Rightarrow$ d: Assume (d) is false, and fix some $P \in$ spec(R) such that $\hat{\Lambda}$ is not irreducible in $\hat{\Gamma}$. Let K be the quotient field of R, and set $\hat{A} = A \otimes_K \hat{K}$, so that $\hat{\Lambda}$ and $\hat{\Gamma}$ are $\hat{R}$-orders in $\hat{A}$. It is no loss to assume that $\hat{A}$ is simple. Since $\hat{\Gamma}$ is a maximal $\hat{R}$-order in the simple algebra $\hat{A}$ and $\hat{R}$ is complete, $\hat{\Gamma}$ has a unique simple module M. By hypothesis, M contains a proper $(\hat{\Lambda}, D)$-submodule, say N, where D is the endomorphism ring of M over $\hat{\Gamma}$. By [3, Proposition 2], $\hat{H} = \{x \in \hat{\Gamma}\colon xN \subseteq N\}$ is a nonmaximal hereditary order in $\hat{\Gamma}$, and clearly $\hat{\Lambda} \subseteq \hat{H}$. It is now easy to construct a nonmaximal hereditary order H such that $\Lambda \subseteq H \subseteq \Gamma$. Thus (c) is false, as desired.

d $\Rightarrow$ a: Let (d) hold. It is enough to assume that A is simple and R is complete. It follows that $J = J(\Gamma)$ is the unique maximal two-sided ideal of Γ. Let Γ' be a maximal order containing Λ, and set $\Lambda' = \Gamma \cap \Gamma'$. Clearly $\Lambda' \supseteq \Lambda$. Since R is complete, (d) implies that Λ, and hence Λ', is irreducible as a subring of Γ, so it follows that $(\Lambda' + J)/J$ is simple. Since J is an invertible Γ-ideal, there exists an integer n such that $J^n\Gamma' \subseteq \Gamma$ but $J^n\Gamma' \not\subseteq J$, and we set $I = J^n\Gamma'$. Notice that I is a left ideal of Γ. Since Γ' is a maximal order, the right order $O_r(I)$ of I is Γ'. But then $\Gamma \cap O_r(I) = \Gamma \cap \Gamma' = \Lambda'$, and therefore I is a two-sided ideal in Λ'. It follows that $(I + J)/J$ is a two-sided ideal in $(\Lambda' + J)/J$, and since the latter is a simple ring, either $I + J = J$ or $I + J = \Lambda' + J$. The first possibility is ruled out since $I \not\subseteq J$, and hence

$$(*) \qquad I + J = \Lambda' + J.$$

Now I is a left ideal of $\Lambda' + J$, and J is contained in the radical of $\Lambda' + J$, so Nakayama's lemma applied to (*) shows that $I = \Lambda' + J$. In particular, $1 \in I$. Inasmuch as I is a left ideal of Γ, we find that $I = \Gamma$, whence $\Gamma = O_r(I) = \Gamma'$, showing that (d) implies (a).

d $\Longrightarrow$ e: This follows directly from the Proposition in §1.

e $\Longrightarrow$ c: Let (e) hold, and assume H is a hereditary R-order such that $\Lambda \subseteq H \subseteq \Gamma$. By assumption, $J(\hat{H}_P) \subseteq J(\hat{\Gamma}_P)$ for all $P \in \operatorname{spec}(R)$. Jacobinski's characterization [3, Theorem 1] of hereditary orders implies that $H = \Gamma$, as desired.

This concludes the proof of the theorem.

It is interesting to single out the following special case:

Corollary. *Let Γ and Γ' be distinct maximal R-orders in A. Then $\Gamma \cap \Gamma'$ is contained in a nonmaximal hereditary order.*

Example. Let K be the field of rational numbers, and let $\mathbf{Z}_{(2)}$ be the localization of the ring $\mathbf{Z}$ of integers at the prime 2. The polynomial $x^2 + x + 1$ is irreducible over the residue class field $\mathbf{Z}/2\mathbf{Z}$ of $\hat{\mathbf{Z}}_{(2)}$, and therefore if Λ is any $\mathbf{Z}_{(2)}$-order in K_2 (the ring of two-by-two matrices over K) which contains the companion matrix $\begin{pmatrix} 0 & 1 \\ -1 & -1 \end{pmatrix}$ of $x^2 + x + 1$, then Λ is unimaximal.

3. **Group algebras.** We use a theorem in modular character theory together with the result of the previous section to characterize those group algebras (with suitable restrictions on the ground ring) which are unimaximal.

A finite group G is said to be p-nilpotent (p a rational prime) if G contains a normal subgroup N of order relatively prime to p such that G/N is a p-group.

Theorem B. *Let G be a finite group of order n, let K be an algebraic number field containing a primitive nth root of unity, and let R be a Dedekind domain with quotient field K. Then RG is unimaximal in KG if and only if*

(1) *G is p-nilpotent, and*

(2) *the Sylow p-subgroups of G are abelian*

for all rational pirmes p which are nonunits in R.

Proof. Let Γ be a maximal R-order in KG which contains $\Lambda = RG$, and fix $P \in \operatorname{spec}(R)$. Let p be the rational prime in P. Since K is a splitting field for KG, it follows that $J(\hat{\Gamma}) = \hat{P}\hat{\Gamma}$. Now let M be an irreducible (in the sense of lattices) left $\hat{\Gamma}$-lattice, so that M is also an irreducible left $\hat{\Lambda}$-lattice (see [5, Chapter IV, Lemma 1.13]). One easily checks that $M/\hat{P}M$ is a simple left $\hat{\Gamma}$-module, and every simple left $\hat{\Gamma}$-module can be obtained in this way. Clearly $\operatorname{End}_{\hat{\Gamma}}(M/\hat{P}M) = \hat{R}/\hat{P}$, so any $(\hat{\Lambda}, \hat{R}/\hat{P})$-bisubmodule of $M/\hat{P}M$ is simply a left $\hat{\Lambda}$-submodule. From Theorem A, we see that $\hat{\Lambda}$ is unimaximal if and only if each such $M/\hat{P}M$ is a simple $\hat{\Lambda}$-module. Noting that $\hat{\Lambda}/\hat{P}\hat{\Lambda} = FG$, where $F = \hat{R}/\hat{P}$, we can apply a theorem of Richen [4] which says that $M/\hat{P}M$ is a simple $\hat{\Lambda}$-module for all such M if and only if G is p-nilpotent and the Sylow p-subgroups are abelian. This concludes the proof.

Corollary. *Let G be a finite group of order n, let K be an algebraic number field containing a primitive nth root of unity, and let R be the ring of algebraic integers in K. Then RG is unimaximal in KG if and only if G is abelian.*

Proof. Note first that every rational prime p is a nonunit in R. From Theorem B we see that RG is unimaximal in KG if and only if G is p-nilpotent with abelian Sylow p-subgroups for all primes p. One can readily show that this is equivalent to G being abelian.

Example. Let $G = S_3$, the (nonabelian) symmetric group on 3 letters, of order 6. Let K be the field of rational numbers, and let $R = \mathbf{Z}_{(3)}$. Since G is not 3-nilpotent and 3 is not a unit in R, RG is not unimaximal. We give an explicit proper hereditary order containing RG. Write $KG = K \oplus K \oplus K_2$, a direct sum of full matrix algebras over K. Now G is generated by the two 2-cycles (12) and (13). The correspondence

$$(12) \to \left(1, -1, \begin{pmatrix} 1 & 0 \\ 1 & -1 \end{pmatrix}\right), \quad (13) \to \left(1, -1, \begin{pmatrix} 1 & 3 \\ 0 & -1 \end{pmatrix}\right)$$

defines an embedding of RG into KG. Now the elements of KG of the form $\left(a, b, \begin{pmatrix} c & 3d \\ e & f \end{pmatrix}\right)$, where $a, b, c, d, e, f \in R$, is a proper hereditary order in KG which clearly contains RG. From this the reader may determine two distinct maximal R-orders in KG which contain RG.

REFERENCES

1. A. Brumer, *Structure of herediatry orders*, Bull. Amer. Math. Soc. 69 (1963), 721–724. MR 27 #2543.

2. M. Harada, *Hereditary orders*, Trans. Amer. Math. Soc. 107 (1963), 273–290. MR 27 #1474.

3. H. Jacobinski, *Two remarks about hereditary orders*, Proc. Amer. Math. Soc. 28 (1971), 1–8. MR 42 #7688.

4. F. Richen, *Decomposition numbers of p-solvable groups*, Proc. Amer. Math. Soc. 25 (1970), 100–104. MR 40 #7356.

5. K. W. Roggenkamp and V. Huber-Dyson, *Lattices over orders*. I, Lecture Notes in Math., vol. 115, Springer-Verlag, Berlin and New York, 1970. MR 44 #247a.

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