PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 52, October 1975

ON THE APPLICATIONS OF COALGEBRAS TO GROUP ALGEBRAS

ALAN ROSENBERG

ABSTRACT. This paper examines the coalgebra structure on $k[G]^\circ$ and relates it to the group theoretic properties of G. In particular it is shown that there is an intimate relation between $k[G]$ being proper and G being residually finite. We use this to derive a series of conditions on the group to guarantee it being residually finite.

It is well known that the dual of a finite dimensional algebra has a canonical coalgebra structure induced on it. This, and the fact that there is a natural isomorphism from $k[G]$ onto $k[G]^*$, when G is finite, has been taken advantage of (cf. [7]) in establishing a coalgebra structure on $k[G]$. The comultiplication is

$$\Delta\sigma = \sum_{\chi\in G} \sigma\chi \otimes \chi^{-1} \quad \forall\sigma\in G$$

and the counit is

$$\epsilon\left(\sum n_\sigma \sigma\right) = n_e$$

where e is the identity element of G. We would like to generalize this to the infinite case.

Unfortunately, the dual of an infinite dimensional algebra is too large to canonically take a coalgebra structure. Therefore, if A is an algebra over k we look at

$$A^\circ = \{f\in A^* \mid \ker f \text{ contains a cofinite ideal}\}.$$

An ideal $J \subseteq A$ is said to be cofinite if A/J is finite dimensional over k. Now there is a canonical way to impose a coalgebra structure on A°. In fact one may view $(\ )^\circ$ and $(\ )^*$ as adjoint functors between algebras and coalgebras.

Let i be the natural mapping from $k[G]$ into $k[G]^*$, given by

$$i\colon \{f\colon G \to k \mid f \text{ has finite support}\} \to \{f\colon G \to k\}.$$

Now several questions arise. What does $k[G]^\circ$ look like? The following theorem is a step in that direction.

Theorem 1. *Let G be an infinite group. Then* $i(k[G]) \cap k[G]^\circ = (0)$.

Received by the editors May 29, 1974 and, in revised form, July 17, 1974.

AMS (MOS) subject classifications (1970). Primary 16A24, 16A26; Secondary 20E15, 20E25.

Key words and phrases. Coalgebra, group algebra, proper, residually finite.

Proof. It can be shown (cf. [6, p. 115]) that $k[G]^*$ is a $k[G]$ module by

$$(1) \qquad (b \rightharpoonup f)(\sigma) = \sum_{\psi} f(\psi) b(\sigma^{-1}\psi), \qquad b \in k[G], \quad f \in k[G]^*, \quad \sigma \in G.$$

Notice that the sum in (1) is finite. If $*$ is the well-known involution on the group algebra and $f \in k[G]$, then equation (1) says that

$$(2) \qquad b \rightharpoonup f = fb^*$$

where the right side of (2) is just ordinary multiplication in $k[G]$. It is well known that $f \in k[G]^*$ will actually be in $k[G]^\circ$ if and only if $k[G] \rightharpoonup f$ is finite dimensional. By (2), this is equivalent to saying that $f \in i(k[G]) \cap k[G]^\circ$ if and only if the right ideal in $k[G]$ generated by f is finite dimensional. But if G is infinite it is easy to see that no right ideal can be finite dimensional. Q.E.D.

Now if A is an algebra then we have a natural mapping, $\rho\colon A \longrightarrow (A^\circ)^*$ as algebras. If ρ is one-to-one we say that A is proper. It can be shown that this is equivalent to every nonzero element of A being disjoint from a cofinite ideal. This raises the question: What are necessary and sufficient conditions on G in order that $k[G]$ be proper?

It can be shown, using Krull's intersection theorem and the Hilbert nullstellensatz, that a commutative finitely generated algebra is proper. Since a finitely generated Abelian group is residually finite the following theorem extends this result in the case of the group algebra.

Theorem 2. *If G is residually finite then $k[G]$ is proper.*

Proof. Let $f \neq 0 \in k[G]$. Consider the set $S = \{\sigma\psi^{-1}\}$ where $\sigma \neq \psi$ runs through supp f. By hypothesis there exists a normal subgroup N of G such that $[G:N]$ is finite and N is disjoint from S. Let π be the natural projection of $k[G]$ onto $k[G/N]$. Since no two elements of supp f are congruent modulo N we clearly have $f \notin \ker \pi$. Yet $k[G]/\ker \pi \approx k[G/N]$ and so $\ker \pi$ is cofinite. Q.E.D.

Hence $k[G]$ is proper for free groups and finitely generated matrix groups over a commutative ring.

The above theorem possesses a partial converse.

Theorem 3. *If $k[G]$ is proper then G is residually finite if either*
(i) k is finite, or
(ii) G is torsion and finitely generated.

Proof. Let δ be any nonidentity element of G. We must exhibit a normal subgroup, N, which is of finite index in G that does not contain δ. Consider the element $\delta - 1$. Since $k[G]$ is proper there exists a cofinite ideal M such that $\delta - 1 \notin M$. Consider the map $\rho\colon G \longrightarrow \mathrm{Aut}_k(k[G]/M)$ defined by $\langle \rho(\iota), f + M\rangle = \iota f + M$ where $\iota \in G$ and $f \in k[G]$. Clearly ρ is a homomorphism to the general linear group $\mathrm{Gl}_n(k)$ where $n = \mathrm{codim}_k M$. Let $N = \ker \rho$. Then N is a normal subgroup of G and it is easy to see that $N = \{\psi \in G \mid \psi - 1 \in M\}$. Hence $\delta \notin N$. Now G/N is a subgroup of $\mathrm{Gl}_n(k)$. If condition (i) prevails then of course $\mathrm{Gl}_n(k)$ and hence also G/N is finite. If case (ii) is true, a theorem of Burnside yields the desired result. (cf. [2]) Q.E.D.

We now proceed to derive sufficient conditions on G for $k[G]$ to be proper.

Theorem 4. *If G is a group and its center Z is both finitely generated and of finite index, then $k[G]$ is proper.*

Proof. Let $\{e = \sigma_0, \sigma_1, \ldots, \sigma_n\}$ be a set of coset representatives for Z in G, where e is the group identity. Since $k[G]$ is a free $k[Z]$ module with $e, \sigma_1, \ldots, \sigma_n$ as a basis, if $f \in k[G]$ is nonzero we may write $f = \sum_{i=0}^{n} h_i \sigma_i$, $h_i \in k[Z]$, and not all the h_i are zero. If $h_i \neq 0$, then since $k[Z]$ is proper, there is a cofinite ideal J_i of $k[Z]$ such that $h_i \notin J_i$. Let $M = \bigcap J_i$. Then M is a cofinite ideal of $k[Z]$ and no nonzero $h_i \in M$. Consider the set $L = \{\sum g_i \sigma_i \mid g_i \in M\}$. Then since the g_i are central we see that L is an ideal. Furthermore, by freeness, $\dim_k k[G]/L = (\dim_k k[G]/M) \cdot ([G:Z])$. Finally, it is clear that $f \notin L$. Q.E.D.

Theorem 5. *If G is a finitely generated torsion-free nilpotent group then $k[G]$ is proper.*

Proof. By a result of Jennings (cf. [3]), the conditions imposed upon G in the hypothesis imply that $\bigcap_n \Delta^n = (0)$ where Δ is the fundamental ideal. Hence our result will follow if we can show that each Δ^n is cofinite. We proceed by induction. Δ is of codimension one. Now assume that Δ^{n-1} is cofinite. Note that Δ/Δ^n is a finitely generated $k[G]/\Delta^{n-1}$ module. Since $k[G]/\Delta^{n-1}$ is itself finite dimensional over k it now follows that so is Δ/Δ^n. But we can now conclude that Δ^n is cofinite.

Theorem 6. *If G is a finitely generated group and has an Abelian subgroup H, of finite index, then $k[G]$ is proper if either*

(i) *characteristic of k is zero, or*

(ii) *characteristic of k is p and G contains no elements of order p.*

Proof. In [5] it is shown that under the assumptions of the hypothesis the simplicial radical of $k[G]$ is trivial. Hence any nonzero element of $k[G]$ is not included in some maximal ideal. But it is well known that the first two conditions on G imply that $k[G]$ is a finitely generated polynomial identity algebra. Theorem 3.3(c) of [4] now guarantees that each maximal ideal is cofinite. Q.E.D.

The previous four theorems combine to yield sufficient conditions for a group to be residually finite.

Theorem 7. *A group G is residually finite if it possesses any of the following properties:*

(i) *The center of G is both finitely generated and of finite index.*

(ii) *G is finitely generated, torsion free and nilpotent.*

(iii) *G is finitely generated, torsion and contains an Abelian subgroup of finite index.*

(iv) *G is finitely generated, has no p torsion elements for some prime p, and contains an Abelian subgroup of finite index.*

(The second condition is not a new result and was proved by Gruenberg (cf. [1]), although in a different manner.)

REFERENCES

1. K. W. Gruenberg, *Residual properties of infinite soluble groups*, Proc. London Math. Soc. (3) 7 (1957), 29–62. MR **19**, 386.

2. I. N. Herstein, *Noncommutative rings*, Carus Math. Monos., no. 15, Math. Assoc. Amer. (distributed by Wiley, New York), 1968. MR **37** #2790.

3. S. A. Jennings, *The group ring of a class of infinite nilpotent groups*, Canad. J. Math 7 (1955), 169–187. MR **16**, 899.

4. Claudio Procesi, *Non commutative Jacobson-rings*, Ann. Scuola Norm. Sup. Pisa (3) **21** (1967), 281–290. MR **37** #251.

5. Alan Rosenberg, *The simplicial radical of the group algebra* (unpublished).

6. Moss Sweedler, *Hopf algebras*, Math. Lecture Notes Ser., Benjamin, New York, 1969. MR **40** #5705.

7. E. J. Taft, *Reflexivity of algebras and coalgebras*, Amer. J. Math. **94** (1972), 1111–1130. MR **46** #9095.

DEPARTMENT OF MATHEMATICS, WESLEYAN UNIVERSITY, MIDDLETOWN, CONNECTICUT 06457