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WEAKLY COMPLETELY CONTINUOUS ELEMENTS OF C^*-ALGEBRAS

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ABSTRACT. For a C^*-algebra A and $u \in A$, the equivalence of the following three statements is proved: (i) the map $x \mapsto uxu$ is a compact operator on A, (ii) (resp. (iii)) the map $x \mapsto ux$ (resp. $x \mapsto xu$) is a weakly compact operator on A. The canonical image of a dual C^*-algebra A in its bidual A^{**} is characterized as the set of the weakly completely continuous elements of A^{**}.

1. **Introduction.** Let E be a Banach space and $L(E)$ the Banach algebra of bounded linear operators on E. K. Vala has proved in [15] that $T \in L(E)$ is a compact operator on E if and only if the map $X \mapsto TXT$ is a compact operator on $L(E)$. Motivated by this phenomenon Vala defined in [16] the element u of an arbitrary Banach algebra A to be *compact*, if the map $x \mapsto uxu$ is a compact operator on A. Subsequent investigations (see [1], [18], [19]) have further indicated that this definition yields indeed a natural extension of the notion of a compact operator.

If H is a Hilbert space, the following nonspatial characterization of the compact operators on H is also available: T. Ogasawara proved in [10] that $T \in L(H)$ is a compact operator if and only if the map $X \mapsto TX$ is a weakly compact operator [6, p. 482] on $L(H)$. In the context of C^*-algebras this result suggests another generalization of the concept of a compact operator. For any Banach algebra A, $u \in A$ is called a *left* (resp. *right*) *weakly completely continuous*—abbreviated l.w.c.c. (resp. r.w.c.c.)—element of A, if the map $x \mapsto ux$ (resp. $x \mapsto xu$) is a weakly compact operator on A. It follows from Corollary 6 in [6, p. 484] that the l.w.c.c. (resp. r.w.c.c.) elements of A form a closed two-sided ideal. In the case of a C^*-algebra these ideals are thus selfadjoint [5, p. 17], and so (as noted by Ogasawara in [10, p. 362]) if $u \in A$ is l.w.c.c. it is also r.w.c.c. (the operator $x \mapsto (u^*x^*)^* = xu$ is weakly compact), and conversely. Therefore we shall simply call the l.w.c.c. (resp. r.w.c.c.) elements of a C^*-algebra *weakly completely continuous* (w.c.c.).

The main result of this paper (Theorem 3.1) states that an element of a C^*-algebra is compact if and only if it is w.c.c., i.e. the two generalizations of a compact operator are in fact the same. The first half of our proof is based on the theorem of Ogasawara mentioned above, but in §2 we give this

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result a short new proof (Corollary 2). We are grateful to the referee for simplifying the second half of the proof of Theorem 3.1.

Let A be a C^*-algebra. We shall regard its second conjugate space A^{**} as a C^*-algebra by identifying it with the enveloping von Neumann algebra of A [5, p. 237]. In [3, p. 869] it is proved that this algebra structure of A^{**} also arises from either one of the two Arens multiplications in A^{**}. If $x \in A$, we let $\tilde{x}$ denote the canonical image of x in A^{**}, and write $\tilde{A} = \{\tilde{x} \mid x \in A\}$.

2. Dual C^*-algebras. It is well known (see [14, p. 533], [8], [2, p. 255]) that a C^*-algebra A is *dual* in the sense of Kaplansky (see e.g. [7]) if and only if $\tilde{A}$ is an ideal (two-sided by selfadjointness) of A^{**}. Several characterizations of dual C^*-algebras are listed in [5, 4.7.20, p. 99]. We note in passing that one further criterion, given in [9, p. 88] (see also [17, p. 538]), follows at once from Theorem 6 in [11, p. 21] (i.e. (v) in [5, p. 99]) and Gantmacher's theorem [6, p. 485].

Theorem 2.1. *Let A be a dual C^*-algebra and $u \in A^{**}$. Then $u \in \tilde{A}$ if and only if u is a w.c.c. element of A^{**}.*

Proof. Suppose first that $u = \tilde{a}$ for some $a \in A$. The definition of the first Arens product in A^{**} (see e.g. [3, p. 848]) shows immediately that the map $x \mapsto ux$ on A^{**} is the second transpose L_a^{**} of $L_a: A \to A$ defined by $L_a x = ax$, $x \in A$. As $\tilde{A}$ is an ideal in A^{**}, $L_a^{**} x = ux \in \tilde{A}$ for all $x \in A^{**}$. Thus L_a is weakly compact [6, p. 482], and so is $L_a^{**}: A^{**} \to A^{**}$ by Gantmacher's theorem [6, p. 485], i.e. u is w.c.c. Suppose, conversely, that u is w.c.c. Since $\tilde{A}$ is an ideal in A^{**}, $u\tilde{A} \subset \tilde{A}$, so the restriction $L = L_u|\tilde{A}$, where $L_u x = ux$, $x \in A^{**}$, may be regarded as an operator from A into itself. We have $L_u = L^{**}$, since both operators are weak* continuous (see [6, p. 478], [3, pp. 848, 869]) and agree on the weak* dense subspace $\tilde{A}$ of A^{**}. Another application of Gantmacher's theorem shows that L is weakly compact so that $L^{**}(A^{**}) \subset \tilde{A}$ [6, p. 482], i.e. $uA^{**} \subset \tilde{A}$. In particular, for the identity 1 of A^{**} we have $u = u1 \in \tilde{A}$.

Corollary 1. *Let A and B be dual C^*-algebras. Any topological algebra isomorphism from A^{**} onto B^{**} maps $\tilde{A}$ onto $\tilde{B}$. In particular, if A^{**} and B^{**} are $*$-isomorphic, then so are A and B.*

Remark. Corollary 1 becomes false, if the word "dual" is omitted. As an example one may consider the C^*-algebra c of all convergent sequences of complex numbers, and its sub-C^*-algebra c_0 consisting of the sequences converging to zero. It is well known that the second conjugate space of both c and c_0 is l^∞, the space of bounded sequences, but c is not isometrically isomorphic to c_0.

Corollary 2 (Ogasawara [10, Theorem 4, p. 362]). *Let H be a Hilbert* *space and* $T \in L(H)$. *Then* T *is a compact operator on* H *if and only if* T *is a w.c.c. element of* $L(H)$.

Proof. Let $C(H)$ denote the ideal of $L(H)$ consisting of all compact operators on H. Since $L(H)$ may be identified with $C(H)^{**}$ in such a way that the canonical embedding of $C(H)$ into $C(H)^{**}$ corresponds to the inclusion map of $C(H)$ into $L(H)$ (see e.g. [5, p. 236] or [13, p. 47]), the corollary is an immediate consequence of the theorem.

3. The equivalence of compactness and weak complete continuity for elements of C^*-algebras. The proof below that (ii) implies (i) is due to the referee. It is considerably shorter than our original argument.

Theorem 3.1. *Let* A *be a* C^**-algebra and* $u \in A$. *The following three conditions are equivalent:*

(i) *the map* $x \mapsto uxu$ *is a compact operator on* A,

(ii) *(resp.* (iii)) *the map* $x \mapsto ux$ *(resp.* $x \mapsto xu$) *is a weakly compact operator on* A.

Proof. It was noted in the introduction that (ii) and (iii) are equivalent. Assume (i). There is an isometric *-representation π of A on a Hilbert space H such that $\pi(u)$ is a compact operator on H [19]. By Corollary 2 in §2, the operators $X \mapsto \pi(u)X$ and $X \mapsto X\pi(u)$ on $L(H)$ are weakly compact. Since $\pi(A)$ is $\sigma(L(H), L(H)^*)$-closed and the relative $\sigma(L(H), L(H)^*)$-topology on $\pi(A)$ agrees with $\sigma(\pi(A), \pi(A)^*)$, it follows that $x \mapsto ux$ and $x \mapsto xu$ are weakly compact operators on A. Assume now (ii). As the ideal W of the w.c.c. elements is selfadjoint, it is a sub-C^*-algebra of A. Since each element of W is w.c.c., W is a dual C^*-algebra by Theorem 6 in [11, p. 21]. As W has an approximate identity [5, p. 15], Cohen's factorization theorem [4, Theorem 1] shows that $u = vw$ for some $v, w \in W$. Thus the operator $x \mapsto uxu$ on A may be written as $T_3T_2T_1$ where $T_1x = xv$, $x \in A$, $T_2y = wyw$, $y \in W$, and $T_3z = vz$, $z \in W$. But $T_2 \colon W \longrightarrow W$ is a compact operator (see e.g. [1, Corollary 8.3]), and so (i) holds.

Note. The compact elements of A form a closed two-sided ideal [18, Theorem 3.10]. This ideal is by Corollary 8.3 in [1] a dual C^*-algebra, whose every element is thus w.c.c. by [11, Theorem 6]. It is therefore clear that the technique used in the second half of the above proof would give an alternate approach to the first half of the proof, too.

Corollary 1. *The* C^**-algebra* A *is dual if and only if its canonical image in* A^{**} *coincides with the closure of the socle of* A^{**}.

Proof. The socle of a Banach algebra is discussed in [12, p. 46]. Since the socle, if it exists, is a two-sided ideal, the condition is sufficient for A to be dual. Suppose now that A is dual. Theorems 2.1 and 3.1 show that

$\tilde{A}$ coincides with the set of the compact elements of A^{**}. But this set is the closure of the socle of A^{**} by Theorems 3.10 and 5.1 in [18].

Of course, Theorem 3.1 transfers all known facts about compact elements of C^*-algebras (for example, the representation theorem of [19]) to the context of w.c.c. elements. In particular, we obtain the following generalization of Ogasawara's theorem (Corollary 2 in §2):

Corollary 2. *Let H be a Hilbert space and A an irreducible sub-C^*-algebra of $L(H)$. Then $T \in A$ is a w.c.c. element of A if and only if T is a compact operator on H.*

Proof. We only need to show that T is a compact operator, if it is a *w.c.c.* element of A. This follows from the above theorem and Corollary 2 in [18, p. 15]. (Note that for C^*-algebras strict and topological irreducibility are equivalent [5, p. 45].)

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