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EXTENDING CONTINUOUS FUNCTIONS IN ZERO-DIMENSIONAL SPACES

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ABSTRACT. Suppose that X is a completely regular, zero-dimensional space and that a dense subset S of X is not C^*-embedded in X; does there then exist a two-valued function in $C^*(S)$ with no continuous extension to X? The answer is negative even if X is a compact space. The question was raised by N. J. Fine and L. Gillman in *Extension of continuous functions in* βN, Bull. Amer. Math. Soc. 66 (1960), 376–381.

This paper answers a question raised by N. J. Fine and L. Gillman in [1]. Suppose that X is a completely regular, zero-dimensional space and that a dense subset S of X is not C^*-embedded in X; does there then exist a two-valued function in $C^*(S)$ with no continuous extension to X? Theorem 1 establishes that the answer is negative.

I. First, I will give some background material, all of which can be found in [2].

All topological spaces are assumed to be completely regular.

The set of all bounded, continuous, real-valued functions on X will be denoted by $C^*(X)$. A subspace S of X is C^*-embedded in X iff every function in $C^*(S)$ can be extended to a function in $C^*(X)$. The Stone-Čech compactification of X is denoted as βX; that is βX is the compactification of X in which X is C^*-embedded.

A space X is *zero-dimensional* if any two completely separated sets in X are contained in complementary open-and-closed sets of X. A space X is zero-dimensional if and only if βX is zero-dimensional.

The space of countable ordinals with the order topology will be denoted by W.

II. **Theorem 1.** *There exists a zero-dimensional space X having a dense subset S such that S is not C^*-embedded in X, but every two-valued function in $C^*(S)$ has a continuous extension to X.*

Proof. Let $I = [0, 1]$ with the usual topology. For each $\alpha \in W$, select $I_\alpha \subset I$ such that I_α is dense in I and $I_\alpha \cap I_\beta = \phi$ if $\alpha \neq \beta$, and such that $\bigcup_{\alpha\in W} I_\alpha = I$.

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Let $S_\alpha = \{\langle x, \alpha\rangle : x \in \bigcup_{\beta\le\alpha} I_\beta\}$ and $S = \bigcup_{\alpha\in W} S_\alpha$. Then $S \subset I \times W$. Topologize S using the relative topology from $I \times W$.

Note that the collection of all neighborhoods of $\langle x, \alpha\rangle$ of the form $\{\langle r, \gamma\rangle : y < r < z$ and $\delta < \gamma \le \alpha\}$ where $y < x < z$ and y and z belong to $\bigcup_{\beta>\alpha} I_\beta$ is a basis of open-closed neighborhoods of $\langle x, \alpha\rangle$ since $\bigcup_{\beta>\alpha} I_\beta$ is dense in I.

Let $X = S \cup \{2\}$. We define a topology on X as follows. S will be an open subspace of X. A neighborhood of 2 is any set U containing 2 such that $2 \in U$ and there is a $\beta \in W$ such that $\{\langle x, \alpha\rangle : \alpha > \beta\} \subset U$.

Since every neighborhood of 2 intersects X, S is dense in X. Also, S is completely regular since $S \subset I \times W$ where both I and W are completely regular.

A consequence of the following proof that X is zero-dimensional is that X is normal. So X is, clearly, completely regular.

To show that X is zero-dimensional, I will show that any two disjoint closed sets in X are contained in complementary open-closed sets. First, consider the case where A and B are disjoint closed sets in X such that $A \cup B \subset C = \bigcup_{\alpha\le\gamma} S_\alpha$, for some $\gamma \in W$. Consider $\alpha_0 \le \gamma$. For each point $\langle x, \alpha_0\rangle \in S_{\alpha_0}$ pick $U(x)$ a basic open-closed neighborhood of $\langle x, \alpha_0\rangle$ such that either $U(x) \cap A$ or $U(x) \cap B$ is empty. Identifying S_{α_0} with $I - \bigcup_{\beta>\alpha_0} I_\beta$, S_{α_0} is second countable, so a countable collection $\{U(x)_n\}_{n\in N}$ covers S_{α_0}. Now, since $\gamma \in W$ there is a countable collection, say $\{V_n\}_{n\in N}$, of open-closed sets, covering C with the property that for each n, either $V_n \cap A$ or $V_n \cap B$ is empty. Define $W_n = V_n - \bigcup_{i<n} V_i$. Then $\{W_n\}_{n\in N}$ is a collection of disjoint open-closed sets which covers C and either $W_n \cap A$ or $W_n \cap B$ is empty. Let $0 = \bigcup\{W_k : W_k \cap A = \phi\}$; then $C - 0 = \bigcup\{W_k : W_k \cap A \ne \phi\}$. So 0 and $C - 0$ are complementary open-closed sets in C and $A \subset C - 0$ and $B \subset 0$. Since $X - C$ is open and closed in X, $0 \cup X - C$ and $C - 0$ are complementary open-closed sets in X.

Now, suppose A and B are disjoint closed sets in X and $2 \in A$. Then there exists a $\beta \in W$ such that $B \subset D = \bigcup_{\alpha\le\beta} S_\alpha$. Since D is closed in X, $A \cap D$ is closed in X. By the above argument there exist complementary open-closed sets H and K in D such that $B \subset H$ and $A \cap D \subset K$. Then H and $K \cup X - D$ are complementary open-closed sets in X such that $B \subset H$ and $A \subset H \cup X - D$. So X is zero-dimensional.

To show that S is not C^*-embedded in X, define $F: S \to I$ by $F(\langle x, \alpha\rangle) = x$. Obviously F is continuous.

However, F cannot be extended continuously to 2, since F assumes all values in every neighborhood of 2.

Every two-valued continuous function on S can be extended continuously to X. Let f be a two-valued continuous function on S with range $\{0, 1\}$. For each $x \in I$, there exists an $\alpha_x \in W$ such that f is constant on $\{\langle x, \beta\rangle: \beta \geq \alpha_x\}$, since for fixed x, the set of all points $\langle x, \alpha\rangle \in S$ is homeomorphic to W.

Now, for each $x \in I$, $x \neq 0, 1$, there exists an integer N_x such that f is constant on

$$U_x = \{\langle y, \alpha\rangle: x - 1/N_x < y < x + 1/N_x, \alpha > \alpha_x\}.$$

If not, then for every integer n, there is a point $\langle y_n, \alpha_n\rangle$ such that $x - 1/n < y_n < x + 1/n$ and $\alpha_n > \alpha_x$ and $f(\langle y_n, \alpha_n\rangle) \neq f(\langle x, \alpha_x\rangle)$. But x is the limit of $\{y_n\}$ and some $\alpha' \in W$ is the limit of $\{\alpha_n\}$, so by the continuity of f, $f(\langle x, \alpha'\rangle) \neq f(\langle x, \alpha_x\rangle)$ which is a contradiction since $\alpha' > \alpha_x$. Similar arguments establish the existence of U_0 and U_1.

For each U_x, consider $U'_x = (x - 1/N_x, x + 1/N_x) \subset I$. The collection $\{U'_x: x \in I\}$ is an open cover of I. Pick a finite subcover $\{U'_{x_i}\}_{i=1}^k$.

Let α_{x_1} be the largest of the ordinals $\{\alpha_{x_i}\}_{i=1}^k$. Then f is constant on $B = \bigcup_{\beta > \alpha_{x_1}} S_\beta$.

Extend f to f': $X \to \{0, 1\}$ by defining $f'(2) = f(B)$. Clearly f' is continuous at 2 since $B \cup \{2\}$ is a neighborhood of 2.

III. **Corollary.** *There exists a zero-dimensional compact space which satisfies Theorem* 1.

Proof. Since X is zero-dimensional, βX is zero-dimensional and S is dense in βX. Since $F \in C^*(S)$ cannot be extended to X, F cannot be extended to βX. But every two-valued function in $C^*(S)$ extends to X and hence to βX. So βX is a compact zero-dimensional space which satisfies Theorem 1.

REFERENCES

1. N. J. Fine and L. Gillman, *Extension of continuous functions in* βN, Bull. Amer. Math. Soc. 66 (1960), 376–381. MR 23 #A619.

2. L. Gillman and M. Jerison, *Rings of continuous functions*, University Ser. in Higher Math., Van Nostrand, Princeton, N. J., 1960. MR 22 #6994.

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