DECOMPOSITION OF TENSOR PRODUCTS
OF IRREDUCIBLE UNITARY REPRESENTATIONS

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ABSTRACT. It is shown that the tensor product of an irreducible unitary representation of a (discrete) group $G$ and an $n$-dimensional ($n < \infty$) unitary representation of $G$ decomposes into at most $n^2$ irreducible subrepresentations; the multiplicity of each irreducible constituent is not greater than $n$. As an application it is shown that the restriction of an irreducible unitary representation to a subgroup of finite index is a finite sum of irreducible subrepresentations.

At the meeting "Harmonische Analyse und Darstellungstheorie lokalkompakter Gruppen" in Oberwolfach, R. Howe posed the following question: Let $G$ be a (discrete) group and let $\pi$ and $\rho$ be irreducible unitary representations of $G$ with $\dim \rho = n < \infty$. Is $\rho \otimes \pi$ a sum of at most $n^2$ irreducible subrepresentations? He proved that the answer is yes if $\pi$ is finite dimensional, too. In this paper it is shown that the answer is yes in the general case. Indeed, we will show a little bit more, namely:

**Theorem.** Let $\pi$ and $\pi'$ be irreducible unitary representations of the group $G$ in the Hilbert spaces $S_\pi$ and $S_{\pi'}$, respectively, and let $\rho$ be a unitary representation of $G$ in a Hilbert space of dimension $n < \infty$. Then the dimension of the space $\text{Hom}_G(\pi', \rho \otimes \pi)$ of intertwining operators is not greater than $n$ and is equal to $n$ iff $\rho \otimes \pi$ is unitarily equivalent to $\pi \otimes \pi'$.

**Corollary 1.** Let $G, \rho, \pi, S_\pi$ be as in the Theorem. Then the dimension of the algebra $\text{Hom}_G(\rho \otimes \pi, \rho \otimes \pi)$ is not greater than $n^2$ and is equal to $n^2$ iff $\rho^* \otimes \rho \otimes \pi$ is unitarily equivalent to $n^2 \pi$ ($\rho^*$ denotes the contragradient representation). Especially, $\rho \otimes \pi$ is the direct sum of at most $n^2$ irreducible subrepresentations.

**Corollary 2.** Let $G, \pi, S_\pi$ be as in the Theorem and let $N$ be a subgroup of $G$ of finite index. Then the restriction of $\pi$ to $N$ is a finite sum of irreducible subrepresentations. If $N$ is a normal subgroup the dimension of the algebra $\text{Hom}_N(\pi, \pi)$ is not greater than the index $[G : N]$.

Corollary 2 was used in the proof of Proposition 2.1 in [1] for locally compact groups of type I.

The basic idea in the proof of the Theorem is the introduction of an inner
product $\langle \, , \rangle$ in $\text{Hom}_G(\rho \otimes \pi, \rho \otimes \pi)$ which is totally algebraic in nature. At the end of the paper, we will give an example of an $n$-dimensional (for all integers $n > 1$) irreducible representation $\rho$ such that $\rho \otimes \rho^*$ is a sum of $n^2$ one-dimensional subrepresentations.

Let $V$ be a finite dimensional Hilbert space and let $\mathcal{H}$ be an arbitrary Hilbert space. The algebraic tensor product $V \otimes \mathcal{H}$ becomes a Hilbert space if we define $\langle w \otimes k, v \otimes h \rangle = \langle w, v \rangle \langle k, h \rangle$. If $e_1, \ldots, e_n$ is an orthonormal basis of $V$, every element in $V \otimes \mathcal{H}$ can be represented uniquely in the form $\sum_{i=1}^{n} e_i \otimes h_i$ and, by definition of the inner product, we get

$$\sum_{i=1}^{n} e_i \otimes h_i$$

Now, we indicate how Corollary 1 (the answer to Howe's question) follows from the Theorem.

Let $\mathcal{H}'$ be another Hilbert space and let $V^*$ denote the dual space of $V$. Then the spaces of bounded operators $\text{Hom}(V \otimes \mathcal{H}, \mathcal{H}')$ and $\text{Hom}(\mathcal{H}, V^* \otimes \mathcal{H}')$ are canonical isomorphic. The isomorphism $J : \text{Hom}(V \otimes \mathcal{H}, \mathcal{H}') \rightarrow \text{Hom}(\mathcal{H}, V^* \otimes \mathcal{H}')$ is given by $(JT)h = \sum_{i=1}^{n} e_i^* \otimes T(e_i \otimes h)$ if $e_1^*, \ldots, e_n^*$ is the dual basis of $e_1, \ldots, e_n$; $JT$ is independent of the basis $e_1, \ldots, e_n$. $J$ is not an isometry for the operator norms, but we have

**Lemma 1.** $n^{-\frac{1}{2}} \|T\| \leq \|JT\| \leq n^{\frac{1}{2}} \|T\|$ for all $T \in \text{Hom}(V \otimes \mathcal{H}, \mathcal{H}')$. More over, let $G$ be a group, and let $\rho, \pi$ and $\pi'$ be unitary representations of $G$ in $V$, $\mathcal{H}$ and $\mathcal{H}'$, respectively. Then $J$ transforms the space of intertwining operators $\text{Hom}_G(\rho \otimes \pi, \pi')$ onto $\text{Hom}_G(\pi, \pi^* \otimes \pi')$.

The simple proof of this lemma is omitted. For $\pi' = \rho \otimes \pi$ we get $\text{Hom}_G(\rho \otimes \pi, \rho \otimes \pi) \cong \text{Hom}_G(\pi, (\rho^* \otimes \rho) \otimes \pi)$ (Theorem $\Rightarrow$ Corollary 1).

Now, for a unitary representation $\rho$ in a finite dimensional Hilbert space $V$ and an irreducible unitary representation $\pi$ in a Hilbert $\mathcal{H}$, we introduce an inner product $\langle \, , \rangle$ in $\text{Hom}_G(\rho \otimes \pi, \rho \otimes \pi)$ which is crucial in the proof of the Theorem (we will use that inner product for different $\rho$'s and $\pi$'s and will denote it by the same symbol $\langle \, , \rangle$). We will compute $\langle TT^*, TT^* \rangle$ for $T \in \text{Hom}_G(\pi, \rho \otimes \pi)$ in two different ways. The inner product is defined as follows:

If $S, T \in \text{Hom}_G(\rho \otimes \pi, \rho \otimes \pi)$ then $JS, JT \in \text{Hom}_G(\pi, (\rho^* \otimes \rho) \otimes \pi)$ and $(JS)^*JT \in \text{Hom}_G(\pi, \pi)$. Since $\pi$ is irreducible, that operator is a scalar multiple of the identity, and we define

$$\langle T, S \rangle \text{id}_\pi = (JS)^*(JT).$$

**Lemma 2.** (i) $\langle \, , \rangle$ is linear in the first and conjugate-linear in the second variable.

(ii) $\langle S, S \rangle = \|J(S)\|^2$, especially, $\langle S, S \rangle = 0$ iff $S = 0$. 


(iii) \( \langle \text{id}_{V \otimes \mathbb{F}}, \text{id}_{V \otimes \mathbb{F}} \rangle = \dim V \).
(iv) \( \langle S, T \rangle = \langle T, S \rangle \).
(v) Let \( e_1, \ldots, e_n \) be an orthonormal basis of \( V \), and let \( S, T \in \text{Hom}_G(\rho \otimes \pi, \rho \otimes \pi) \) be given by

\[
S(e_i \otimes h) = \sum_{k=1}^n e_k \otimes S_{ki}(h) \quad \text{and} \quad T(e_i \otimes h) = \sum_{k=1}^n e_k \otimes T_{ki}(h)
\]

for \( h \in \mathbb{F} \) and \( 1 \leq i \leq n \). Then the equation \( \langle T, S \rangle = \sum_{i,k=1}^n S_{ki}^* T_{ik} \) holds.

Proof. (i) and (ii) are clear, (iii) and (iv) follow from (v). Let \( e_1, \ldots, e_n \) be an orthonormal basis of \( V \), and let \( e_1^*, \ldots, e_n^* \) be the dual basis of \( e_1, \ldots, e_n \). By definition, we have \( (JS)(h) = \sum_{i,k=1}^n e_i^* \otimes e_k \otimes S_{ki}(h) \). A simple computation shows that

\[
(JS)^* = \sum_{i,k=1}^n S_{ki}^* h_k i
\]

and therefore,

\[
(JS)^*(JT) = \sum_{i,k=1}^n S_{ki}^* T_{ik}.
\]

Definition. Let \( \rho \) and \( \pi \) be as before. Let \( \pi' \) be another irreducible unitary representation of \( G \) in \( \mathbb{F}' \) and let \( T \in \text{Hom}_G(\pi', \rho \otimes \pi) \). The operator \( T \in \text{Hom}_G(\rho \otimes \pi, \pi') \) corresponds to an operator in \( \text{Hom}_G(\pi', \rho^* \otimes \pi') \) (by Lemma 1). This operator is denoted by \( Ta \). Explicitly: if \( e_1, \ldots, e_n \) is an orthonormal basis of \( V \) with dual basis \( e_1^*, \ldots, e_n^* \) and \( T \) is represented as

\[
T h_i = \sum_{i=1}^n e_i \otimes T_i h', \quad \text{then} \quad T^a h = \sum_{i=1}^n e_i^* \otimes T_i^a h.
\]

Lemma 3. Let \( 0 \neq T \in \text{Hom}_G(\pi', \rho \otimes \pi) \). Then \( T^a \in \text{Hom}_G(\pi', \pi') \), \( (T^a)^* T^a \in \text{Hom}_G(\rho, \pi) \), \( T^a T^a^* \in \text{Hom}_G(\rho \otimes \pi', \rho^* \otimes \pi') \). Let the positive real numbers \( \alpha \) and \( \beta \) be defined by \( T^a = \alpha \text{id}_{\mathbb{F}'} \), and \( (T^a)^* T^a = \beta \text{id}_{\mathbb{F}'} \) respectively (\( \pi' \) and \( \pi \) are irreducible). Then one has

\[
\langle T^a(T^a)^* \rangle = \alpha \beta = \langle T^a(T^a)^* \rangle.
\]

Moreover, let \( S : \mathbb{F}' \rightarrow V \otimes \mathbb{F} \) be another intertwining operator such that the ranges of \( S \) and \( T \) are orthogonal. Then one has \( \langle SS^*, TT^* \rangle = 0 = \langle S^a(S^a)^*, T^a(T^a)^* \rangle \).

Proof. The first statements are clear. Let \( e_1, \ldots, e_n \) be an orthonormal basis of \( V \) and let \( T \) be given by \( T h_i = \sum_{i=1}^n e_i \otimes T_i h' \). Then

\[
T^a T = \sum_{i=1}^n T^a_i T_i = \alpha \text{id}_{\mathbb{F}'}, \quad (T^a)^* T^a = \sum_{i=1}^n T_i^* T_i = \beta \text{id}_{\mathbb{F}'}
\]

and
By Lemma 2(v), we get

$$\langle\langle TT^*, TT^* \rangle\rangle id_\Phi^* = \sum_{i,k=1}^n (T_i^* T_k^*)^* S_i^* S_k^* = \sum_{i,k=1}^n T_i^* T_k^* S_i^* S_k^* = \alpha \beta id_\Phi^*.$$ 

Since $(T^a)^a = T$ the same argument shows that $\alpha \beta = \langle\langle T^a(T^a)^a, T^a(T^a)^a \rangle\rangle$. Let $S$ be as in the lemma. From $T(S_i) \perp S(S_i)$ we get $T^* S = 0$ and, therefore,

$$(1) \quad \sum_{i=1}^n T_i^* S_i = 0.$$ 

Since $(T^a)^a S^a \in \text{Hom}_G(\pi, \pi)$ there exists $\gamma \in C$ such that

$$(2) \quad \gamma id_\Phi^* = (T^a)^a S^a = \sum_{i=1}^n T_i S_i^*.$$ 

By Lemma 2(v):

$$\langle\langle S^a, TT^* \rangle\rangle id_\Phi^* = \sum_{i,k=1}^n (T_i^* T_k^*)^* S_i^* S_k^* = \sum_{i,k=1}^n T_i^* T_k^* S_i^* S_k^* = 0 \quad (by \ (1)).$$

Now, $T^a(T^a)^a$ (and, analogously, $S^a(S^a)^a$) is given by $T^a(T^a)^a(e_i^* \otimes h') = \sum_{k=1}^n e_i^* \otimes T_k^* T_i h'$. Lemma 2(v) implies:

$$\langle\langle S^a(S^a)^a, T^a(T^a)^a \rangle\rangle id_\Phi^* = \sum_{i,k=1}^n (T_i^* T_k^*)^* S_i^* S_k^* = \sum_{i,k=1}^n T_i^* T_k^* S_i^* S_k^* = 0 \quad (by \ (2) \ and \ (1)).$$

Now we are able to give the

**Proof of the Theorem.** Let $T_1, \ldots, T_r$ be $r$ linearly independent elements in $\text{Hom}_G(\pi, \rho \otimes \pi)$. Without loss of generality we may assume that $T_i(S_i') \perp T_j(S_j')$ for $i \neq j$ (by orthogonalization with respect to the inner product $(S, T) id_\Phi^* = T^* S$). We have to show that $r \leq n$. Let $\alpha_i, \beta_i, P_i$ and $Q_i$ ($1 \leq i \leq r$) be defined by $T_i^* T_i = \alpha_i id_\Phi^*, (T^a_i)^a T_i = \beta_i id_\Phi^*, P_i = \alpha_i^{-1} T_i^* T_i$ and $Q_i = \beta_i T_i^a(T_i^a)^a$. Then the $P_i$'s and $Q_i$'s are projections in $\text{Hom}_G(\rho \otimes \pi, \rho \otimes \pi)$ and $\text{Hom}_G(\rho^* \otimes \pi^*, \rho^* \otimes \pi^*)$, respectively. By Lemma 3, one has

$$\langle\langle P_i, P_j \rangle\rangle = \beta_i \alpha_j^{-1}, \quad \langle\langle Q_i, Q_j \rangle\rangle = \alpha_i \beta_j^{-1},$$

and

$$\langle\langle Q_i, Q_j \rangle\rangle = 0 = \langle\langle P_i, P_j \rangle\rangle \quad for \quad i \neq j.$$ 

Let $P$ and $Q$ be defined by the equations $id_{V \otimes \Phi^*} = P + \sum_{i=1}^r P_i$ and $id_{V^* \otimes \Phi^*} = Q + \sum_{i=1}^r Q_i$. From Lemma 2, one can easily conclude that

$$\langle\langle id_{V \otimes \Phi^*}, P_i \rangle\rangle = \langle\langle P_i, P_i \rangle\rangle = \langle\langle P_i, id_{V \otimes \Phi^*} \rangle\rangle$$ 

and
DECOMPOSITION OF TENSOR PRODUCTS OF REPRESENTATIONS

$$\langle \text{id}_{V \otimes \mathcal{F}}, \mathcal{O}_i \rangle = \langle \mathcal{O}_i, \mathcal{O}_i \rangle = \langle \mathcal{O}_i, \text{id}_{V \otimes \mathcal{F}} \rangle$$

(because the $P_i$'s and $Q_i$'s are projections). This shows that $\langle \langle P_j, P \rangle \rangle = 0 = \langle \langle Q_j, Q \rangle \rangle$ for $1 \leq j \leq r$. Lemma 2 and the above relations imply

$$\langle P, P \rangle + \sum_{i=1}^{r} \beta_i \alpha_i^{-1} = n = \langle Q, Q \rangle + \sum_{i=1}^{r} \alpha_i \beta_i^{-1}.$$  

Since $\langle , \rangle$ is positive definite one has $2n \geq \sum_{i=1}^{r} (\beta_i \alpha_i^{-1} + \alpha_i \beta_i^{-1})$. From $x + x^{-1} \geq 2$ for positive real numbers $x$ we get $r \leq n$ as desired.

If the dimension of $\text{Hom}_G(\pi', \rho \otimes \pi)$ is equal to $n$, choose $n$ linearly independent elements $T_1, \ldots, T_n$ in that space with $T_i(\mathcal{O}_j) = T_j(\mathcal{O}_i)$ for $i \neq j$ and form $P_1$ and $P$ as above. It is easy to see that $\langle \langle P, P \rangle \rangle = 0$ and, therefore, $P = 0$ or $\text{id}_{V \otimes \mathcal{F}} = \sum_{i=1}^{r} \alpha_i^{-1} T_i$. This shows that $V \otimes \mathcal{O}_i$ is the (orthogonal) sum of the $T_i(\mathcal{O}_i)$'s but the restriction of the representation $\rho \otimes \pi$ to $T_i(\mathcal{O}_i)$ is unitarily equivalent to $\pi'$. The "if-part" is trivial. The Theorem is proved, we have already pointed out how Corollary 1 follows from the Theorem.

**Remark.** The fact that $\rho \otimes \pi$ is the direct sum of at most $n^2$ irreducible subrepresentations can be proved quicker if one uses Lemma 1 and a similar trick as in the proof of the Theorem. More precisely, let $P_1, \ldots, P_r$ be orthogonal projections in $\text{Hom}_G(\rho \otimes \pi, \rho \otimes \pi)$. As in the proof of the Theorem one gets $n \geq \sum_{i=1}^{r} \langle \langle P_i, P_i \rangle \rangle$. But $\langle \langle P_i, P_i \rangle \rangle = \| f(P_i) \|^2$ and $\| f(P_i) \| \geq n^{-\frac{1}{2}} \| P_i \| = n^{-\frac{1}{2}}$ (Lemma 1) and, therefore, $n \geq \frac{r}{n}$ or $r \leq n^2$. Of course, the Theorem gives a more precise description.

**Proof of Corollary 2.** Since $\bigcap_{g \in G} gN^{-1}$ is a normal subgroup of finite index, it suffices to prove the second statement. The group $G$, resp. $H := G/N$, acts linearly on the space $\text{Hom}_N(\pi, \pi)$ by $g \cdot f = \pi(g)f \pi(g)^*$. Let $\rho$ be any irreducible unitary representation of $H$ in the finite dimensional Hilbert space $E$; we consider $\rho$ as a representation of $G$, too. The space of intertwining operators $\text{Hom}_H(\rho, \text{Hom}_N(\pi, \pi)) = \text{Hom}_G(\rho, \text{Hom}_N(\pi, \pi))$ is isomorphic to $\text{Hom}_G(\rho \otimes \pi, \pi)$. By the Theorem, the dimension of that space is not greater than $\dim E$. Since $H$ is finite, every element in $\text{Hom}_N(\pi, \pi)$ is contained in a finite dimensional $H$-invariant subspace of $\text{Hom}_N(\pi, \pi)$. Therefore, the dimension of $\text{Hom}_N(\pi, \pi)$ is not greater than $\sum_\rho (\dim \rho)^2$, $\rho$ being an equivalence class of irreducible unitary representations of $H$. But, by a well-known theorem in the representation theory of finite groups, the value of this sum is exactly the order of $H$.

**Example.** For all integers $n > 1$ we will give an example of an $n$-dimensional irreducible representation $\rho$ and another irreducible representation $\pi$ such that $\rho \otimes \pi$ decomposes into exactly $n^2$ subrepresentations. To motivate our example let $\rho$ and $\pi$ be such representations, $\rho \otimes \pi = \bigoplus_{i=1}^{n^2} \pi_i$.

Then the algebra $\text{Hom}_G(\rho \otimes \pi, \rho \otimes \pi)$ is at least $n^2$ dimensional; from Corollary 1 we know that its dimension is exactly $n^2$. Therefore, the $\pi_i$'s are uni-
tarily nonequivalent. Moreover, again by Corollary 1, \( \rho^* \otimes \rho \otimes \pi \) is unitarily
equivalent to \( n^2 \pi \); but on the other hand \( \rho^* \otimes \rho \otimes \pi \) is equal to \( \rho^* \otimes (\bigoplus_{i=1}^{n^2} \pi_i) \). Hence \( \pi \) is unitarily equivalent to \( \rho^* \otimes \pi_i \) for all \( i \).

Let \( G \) be the Heisenberg group over \( \mathbb{Z}/n\mathbb{Z} \),

\[
G = \left\{ \begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} : x, y, z \in \mathbb{Z}/n\mathbb{Z} \right\}.
\]

The commutator subgroup \( G' \) is equal to the center

\[
ZG = \left\{ \begin{pmatrix} 1 & 0 & z \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} : z \in \mathbb{Z}/n\mathbb{Z} \right\}.
\]

Let \( \chi \) be a character of

\[
N := \left\{ \begin{pmatrix} 1 & 0 & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} : y, z \in \mathbb{Z}/n\mathbb{Z} \right\}
\]

which is faithful on \( ZG \) and let \( \rho \) be the induced representation \( \text{ind}_{N \to G} \chi \)
in the \( n \)-dimensional space \( V \). \( \rho \) is irreducible by the Frobenius reciprocity
theorem: the restriction of \( \rho \) to \( N \) decomposes into \( n \) different characters
since \( \chi \) is faithful on \( ZG \). Choose \( \pi = \rho^* \). For \( g \in ZG \) we have

\[
\rho(g) = \chi(g) \text{id}_V, \quad \pi(g) = \rho^*(g) = \chi(g) \text{id}_{V^*},
\]

and therefore,

\[
(\rho \otimes \pi)(g) = \chi(g) \text{id}_V \otimes \overline{\chi(g)} \text{id}_{V^*} = \text{id}_V \otimes \text{id}_{V^*}.
\]

Hence the homomorphism \( \rho \otimes \pi \) factors through \( G/ZG \) which is abelian, and
\( \rho \otimes \pi \) decomposes into one-dimensional subrepresentations; \( \rho \otimes \pi \) is the sum
of all \( n^2 \) nonequivalent (see above) one-dimensional representations of \( G \).

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REFERENCE


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