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A GENERAL HOFFMAN-WERMER THEOREM FOR ALGEBRAS OF OPERATOR FIELDS

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ABSTRACT. Let A be a closed separating subalgebra of $C(T)$ that contains the identity. It is known that $\operatorname{Re} A$ is uniformly closed only if $A = C(T)$. In this note it is shown that this property characterizes all maximal full algebras of operator fields and not just $C(T)$.

1. Introduction. Let T be a compact Hausdorff space and $C(T)$ the space of all complex-valued continuous functions on T. It is of interest and importance to know which properties of $C(T)$ cannot be shared with any of its proper separating subalgebras. In other words, which properties of $C(T)$ characterize it. A classic example of such a property, due to Hoffman and Wermer [3], is the following: If A is a closed separating subalgebra of $C(T)$ that contains the identity, then $\operatorname{Re} A$ is uniformly closed only if $A = C(T)$. It is the purpose of this note to extend the Hoffman-Wermer theorem to the setting of a maximal full algebra of operator fields. Thus, it shows that the Hoffman-Wermer property characterizes all maximal full algebras of operator fields and not just $C(T)$.

We will now define a maximal full algebra of operator fields on T. For each $t \in T$, let C_t be a C^*-algebra with identity I_t. For different values of t the C_t are in general unrelated. An operator field (with respect to $\{C_t\}_{t\in T}$) is a function x on T such that $x(t) \in C_t$ for each $t \in T$. By a full algebra of operator fields on T, we mean a set $\mathrm{C}(T)$ of operator fields x on T that satisfy the following: (i) $\mathrm{C}(T)$ is a $*$-algebra under pointwise algebraic operations; (ii) for each function x in $\mathrm{C}(T)$, the function $t \to \|x(t)\|$ is continuous; (iii) for each $t \in T$, $\{x(t): x \in \mathrm{C}(T)\}$ is dense in C_t; (iv) $\mathrm{C}(T)$ is complete in the norm $\|x\| = \sup\{\|x(t)\|: t \in T\}$. Clearly, $\mathrm{C}(T)$ is a C^*-algebra; hence (iii) could be strengthened to the statement that $\{x(t): x \in \mathrm{C}(T)\} = C_t$. The algebra C_t will be called the component of $\mathrm{C}(T)$ at t. An operator field x is continuous at t_0 (with respect to $C(T)$) if for each $\epsilon > 0$ there is an element y of $\mathrm{C}(T)$ and a neighborhood U of t_0 such that $\|x(t) - y(t)\| < \epsilon$ for all $t \in U$. We say that x is continuous on T if it is continuous at each point of T. It is well known that $\mathrm{C}(T)$ is a maximal full algebra of operator fields

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if and only if it contains all operator fields x which are continuous on T with respect to $\mathrm{C}(T)$. For basic concepts and results on maximal full algebras of operator fields, we refer the reader to [1, Chapter 10] or [2].

2. **The general Hoffman-Wermer theorem.** Throughout this section $\mathrm{C}(T)$ will denote a maximal full algebra of operator fields. We shall assume that the operator field $t \longrightarrow I_t$ belongs to $\mathrm{C}(T)$, which we will denote by I. Let A be a subalgebra of $\mathrm{C}(T)$. Each $x \in A$ has a unique decomposition $x = h_1 + ih_2$, where h_1 and h_2 are hermitian elements of $\mathrm{C}(T)$. We will let $\mathrm{Re}\, x$ denote h_1 and $\mathrm{Im}\, x$ denote h_2. We will let $\mathrm{Re}\, A = \{\mathrm{Re}\, x \colon x \in A\}$ and $A^h = \{x \in A \colon x \text{ hermitian}\}$. Note that $\mathrm{Re}\, A$ and A^h are linear subspaces of A. Moreover, when A is uniformly closed, so is A^h. For $x \in A$ we denote by $\sigma(x)$ and $\sigma_A(x)$ the spectrum of x with respect to $\mathrm{C}(T)$ and A, respectively. Similarly, we define $\sigma_{C_t}(x(t))$ for each $t \in T$. If ξ is a normal element in C_t and f is a complex-valued continuous function on $\sigma_{C_t}(\xi)$, then we let $f(\xi)$ denote the element in C_t defined in the usual way for C^*-algebras.

2.1. **Lemma.** *Let A be a uniformly closed subalgebra of $\mathrm{C}(T)$ and x an element of A. Then the following statements are true:*

(1) *if $x \in A^h$ and f is a real-valued continuous function on $\sigma(x) \cup \{0\}$ with $f(0) = 0$, then the operator field $t \longrightarrow f(x(t))$, denoted $f(x)$, belongs to A^h;*

(2) *if x is normal and f is an analytic function in some simply connected region containing $\sigma(x) \cup \{0\}$ with $f(0) = 0$, then the operator field $t \longrightarrow f(x(t))$ denoted $f(x)$, belongs to A.*

Proof. The proof follows immediately from the fact that f in both cases can be uniformly approximated by polynomials of the form $a_1\lambda + a_2\lambda^2 + \cdots + a_n\lambda^n$.

2.2. **Definition.** A subalgebra A of $\mathrm{C}(T)$ is said to satisfy the H-W condition if the following statements hold: (1) $\mathrm{Re}\, A$ is uniformly closed; (2) for each nonsingleton compact subset F of T and $t_0 \in F$ there is a $t_1 \in F$ and a normal element x in A such that $x(t_0) = 0$, $x(t_1) = I_t$, and $\sup\{\|x(t)\| \colon t \in F\} = 1$.

Note that in the function algebra setting of Hoffman and Wermer the algebra A separates points of T if and only if condition (2) of 2.2 holds. Consequently, the following result extends the Hoffman-Wermer theorem to the setting of a maximal full algebra of operator fields. Furthermore, it extends [1, 10.4.6, p. 197] and partially extends [1, 11.5.3, p. 234] to subalgebras of $\mathrm{C}(T)$ that are not necessarily selfadjoint.

2.3. **Theorem (Hoffman-Wermer).** *Suppose A is a uniformly closed subalgebra of $\mathrm{C}(T)$ that satisfies the H-W condition. Then for each complex-valued continuous function f on T the operator field $t \longrightarrow f(t)I_t$ belongs to A. Moreover, if $A_t = C_t$ for each $t \in T$, then $A = \mathrm{C}(T)$.*

Proof. Let $t_0 \in T$ and U the set of all $t \in T$ for which there is an x in A^h such that $x(t) = I_t$ and $x(t_0) = 0$. Suppose there exists a $t_1 \in T$ and an $x \in A^h$ such that $\sigma_{C_{t_0}}(x(t_0)) \cup \{0\}$ and $\sigma_{C_{t_1}}(x(t_1))$ are disjoint. Let V_0 and V_1 be open neighborhoods of $\sigma_{C_{t_0}}(x(t_0)) \cup \{0\}$ and $\sigma_{C_{t_1}}(x(t_1))$, respectively, with disjoint closures. By [1, 10.3.6, p. 195] there exist open neighborhoods W_0 and W_1 of t_0 and t_1, respectively, such that $\sigma_{C_t}(x(t)) \subseteq V_0$ for all $t \in W_0$ and $\sigma_{C_t}(x(t)) \subseteq V_1$ for all $t \in W_1$. By Urysohn's lemma there is a real-valued continuous function p such that $0 \leq p(\lambda) \leq 1$ for all real λ, $p(\lambda) = 0$ for all λ in V_0, and $p(\lambda) = 1$ for all λ in V_1.

By 2.1, $p(x) \in A$ and it follows that $p(x)(t) = p(x(t)) = 0$ for $t \in W_0$ and $p(x)(t) = p(x(t)) = I_t$ for $t \in W_1$. Hence U is open and equal to the set of all $t \in T$ for which there is an x in A^h such that $\sigma_{C_t}(x(t))$ and $\sigma_{C_{t_0}}(x(t_0)) \cup \{0\}$ are disjoint. Actually, our argument tells us more. Let K be a compact subset of U. By adapting our argument we can show that there is a positive element x in the unit ball of A and a neighborhood V of t_0 such that $x(t) = I_t$ for $t \in K$ and $x(t) = 0$ for $t \in V$.

Now let $F = T \setminus U$. Suppose $F \neq \{t_0\}$. By virtue of the H-W condition there is a $t_1 \in F$ and a normal element y in A such that $y(t_0) = 0$, $y(t_1) = I_{t_1}$, and $\sup\{\|y(t)\|: t \in F\} = 1$. Since $\operatorname{Re} A$ is uniformly closed, we have, by virtue of the open mapping theorem, a positive integer n for which the following statement holds: if $u \in \operatorname{Re} A$ with $\|u\| < 1$, then there exists an $x \in A$ such that $\operatorname{Re} x = u$ and $\|x\| < n$. Let

$$\eta = [\exp(2n + 1)\pi + 1]/[\exp(2n + 1)\pi - 1],$$

and let D_η and R be the regions in the complex plane defined by $D_\eta = \{\lambda: |\lambda| < \eta\}$ and $R = \{\lambda: -\tfrac{1}{2} < \operatorname{Im} \lambda < \tfrac{1}{2}\}$. By using simple combinations of well-known conformal maps it is easy to deduce that the function g defined on D_η by the formula

$$g(\lambda) = [1/\pi][\ln(1 + \lambda/\eta) - \ln(1 - \lambda/\eta)]$$

maps D_η onto R. Moreover, $g(0) = 0$, $g(1) = 2n + 1$, and the Taylor series expansion of g is given by

$$g(\lambda) = [2/\pi] \sum_{k=0}^{\infty} [1/(2k + 1)][\lambda/\eta]^{2k+1}.$$

Since the coefficients of the Taylor series expansion are real and the convergence is uniform on $\overline{D}_1 = \{\lambda: |\lambda| \leq 1\}$, there exists a positive integer m and a positive number α such that the function h defined on $\overline{D}_\lambda$ by the formula

$$h(\lambda) = [2\alpha/\pi] \sum_{k=0}^{m} [1/(2k + 1)][\lambda/\eta]^{2k+1}$$

has the following properties: $h(0) = 0$; $h(1) = 2n + 1$; $-1 < \operatorname{Im} h(\lambda) < 1$ for all $\lambda \in \overline{D}_1$. Let $K = \{t \in T: \|\operatorname{Im} h(y)(t)\| \geq 1\}$. Since y is normal, it follows from 2.1 that K is a compact subset of U. Therefore there is a positive z in the unit ball of A such that $z(t) = I_t$ for $t \in K$ and $z(t_0) = 0$. Set

$$w_1 = (I - z)(h(y))(I - z) + (2n + 1)(2z - z^2).$$

It is easy to verify that $w_1(t_0) = 0$, $w_1(t_1) = (2n + 1)I_{t_1}$, and $\|\operatorname{Im} w_1\| < 1$. As remarked earlier, we have by the open mapping theorem an element w_2 in A such that $\operatorname{Re} w_2 = \operatorname{Re} iw_1$ and $\|w_2\| \leq n$. Set $x = w_1 + iw_2$, which clearly belong to A^h. Since

$$\sigma_{C_{t_1}}(x(t_1)) = 2n + 1 + \sigma_{C_{t_1}}(iw_2(t_1)) \subseteq [n + 1, 3n + 1]$$

and

$$\sigma_{C_{t_0}}(x(t_0)) = \sigma_{C_{t_0}}(iw(t_0)) \subseteq [-n, n],$$

it follows that $t_1 \in U$, which is a contradiction. Hence $T \backslash U = \{t_0\}$.

Let K_1 and K_2 be pairwise disjoint compact subsets of T. Since t_0 was chosen arbitrarily, it is straightforward to show there exists a positive element x in the unit ball of A such that $x(t) = I_t$ for $t \in K_1$ and $x(t) = 0$ for $t \in K_2$. Now let f be a complex-valued continuous function on T. We wish to show that the operator field $t \longrightarrow f(t)I_t$ belongs to A. Clearly, we may assume f is positive and $\|f\|_\infty = 1$. Let $\epsilon > 0$.

Choose a positive integer n so that $5/n < \epsilon$. For each integer j, $1 \leq j \leq n$, set

$$F_j = \{t \in T: (j - 1)/n \leq f(t) \leq j/n\}.$$

It is clear that $\{F_j\}_{j=1}^n$ is a collection of compact sets such that $\bigcup_{j=1}^n F_j = T$. Now for each j define the open set

$$V_j = \{t \in T: (2j - 3)/2n < f(t) < (2j + 1)/2n\}$$

which clearly contains F_j. We now know there are positive elements $\{z_j\}$ in the unit ball of A such that $z_j(t) = I_t$ for $t \in F_j$ and $z_j(t) = 0$ for $t \in T \backslash V_j$. Since each $t \in T$ belongs to some F_j, $z(t) \geq z_j(t) = I_t$, where $z = \sum_{k=1}^n z_k$. Thus $\sigma_{C_t}(z(t)) \geq 1$ for all $t \in T$. By virtue of 2.1 it is easy to see that the operator field $t \longrightarrow z(t)^{-1}$ belongs to A. Set $x_j = z^{-1/2} z_j z^{-1/2}$ for $j = 1, 2, \ldots, n$. Let $t \in T$. Then there is an F_j such that $t \in F_j$. Moreover, the element t must belong to V_j and possibly V_{j-1} or V_{j+1}, but no other open set from our collection $\{V_k\}_{k=1}^n$. Note that $\sum_{k=1}^n x_k = I$, so

$$\left\| f(t)I_t - \sum_{k=1}^{n} \frac{k-1}{n} x_k(t) \right\| = \left\| \sum_{k=1}^{n} \left[f(t) - \frac{k-1}{n} \right] x_k(t) \right\|$$

$$\leq \sum_{k=j-1}^{j+1} \left| f(t) - \frac{k-1}{n} \right| \leq \frac{5}{n} < \epsilon.$$

Thus the operator field $t \longrightarrow f(t)I_t$ belongs to A since A is uniformly closed.

Finally we wish to show $A = \mathrm{C}(T)$ if $A_t = C_t$ for each $t \in T$. Note that A separates points of T in the sense of [4, p. 177] and that each closed subset of T is an intersection of peak sets in the sense of [4, Definition 4.5, p. 181]. So by [4, Theorem 4.6, p. 182], [4, Corollary 4.2, p. 179], and [4, Theorem 4.15, p. 186], $A = \mathrm{C}(T)$.

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